Momentum space topology, Anomalous Quantum Hall Effect, and problems with the Equilibrium Chiral magnetic effect

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&

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M.A.Zubkov, « Absence of equilibrium chiral magnetic effect » arXiv:1605.08724, **Physical Review D 93, 105036 (2016)**

M.A.Zubkov, « Wigner transformation, momentum space topology, and anomalous transport » arXiv:1603.03665, submitted to Annals of Physics. *The positive answer from the editor has been received, minor corrections are requested*

Main points

1. Wigner transform in compact momentum space

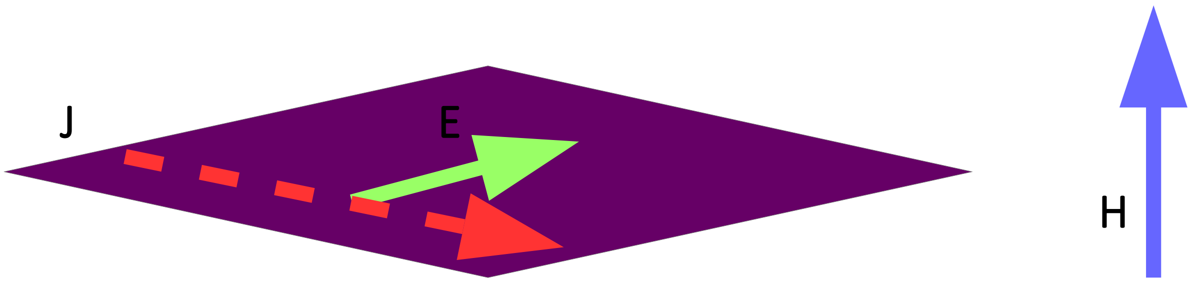
2. Derivative expansion applied to the Wigner transform of the two - point Green function ==>

J = [top.invariant in momentum space]x[field strength]

3. AQHE in 2+1 D, 3+1 D topological insulators, and in 3+1 D Weyl semimetals

4. equilibrium static bulk CME does not exist because the corresponding top. Invariant = 0

Hall effect is the appearance of electric current in the direction orthogonal to the external magnetic field and external electric field.

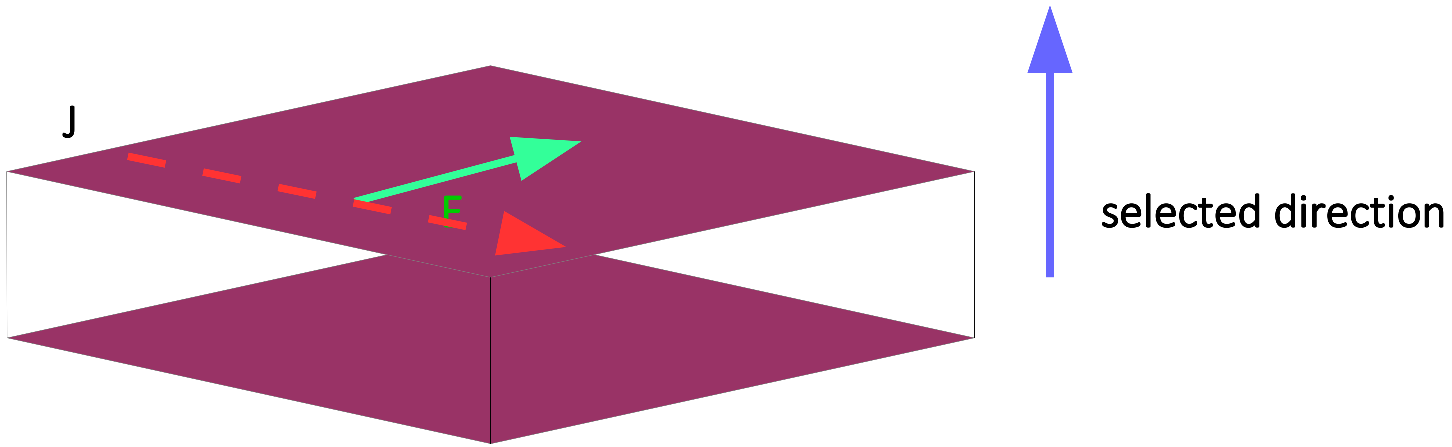


Qauntum Hall effect is the quantized Hall effect

J = N/2pi E

Anomalous Quantum Hall (AQHE) effect is the appearance of quantized current orthogonal to electric field without any external magnetic field (experimentally discovered in 2D materials).

AQHE Hall effect is the appearance of electric current in the direction orthogonal to the external electric field.



J = M /4pi^2 E

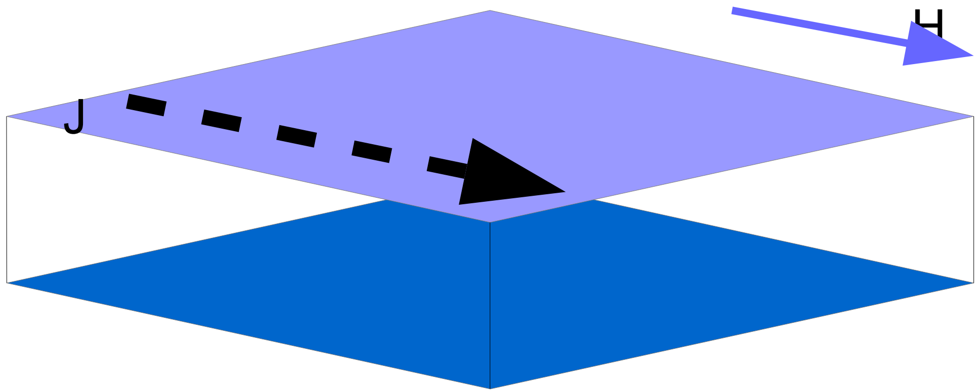
Weyl semimetals

M = [distance between the Weyl points in momentum space]

Topological insulators

M = const/a = integer x [vector of inverse lattice]

Chiral Magnetic Effect (CME) is the appearance of electric current in the direction of the external magnetic field in the presence of chiral chemical potential



J = M /2pi^2 H   M = mu_5 ?

Pre – history: the existence of chiral magnetic effect was proposed in

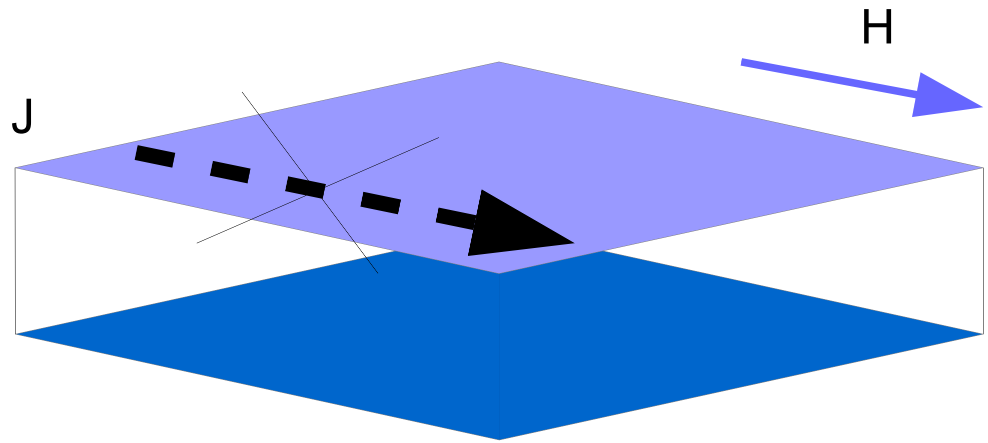
A. Vilenkin, Equilibrium parity-violating current in a magnetic field, Phys. Rev. D **22**, 3080 (1980).

This proposition was later repeated in

K. Fukushima, D. E. Kharzeev, and H. J. Warringa, Chiral magnetic effect, Phys. Rev. D **78**, 074033 (2008).

and in the sequence of the other papers

Chiral Magnetic Effect (CME) is the appearance of electric current in the direction of the external magnetic field in the presence of chiral chemical potential



Later the existance of the equilibrium bulk static CME was questioned.

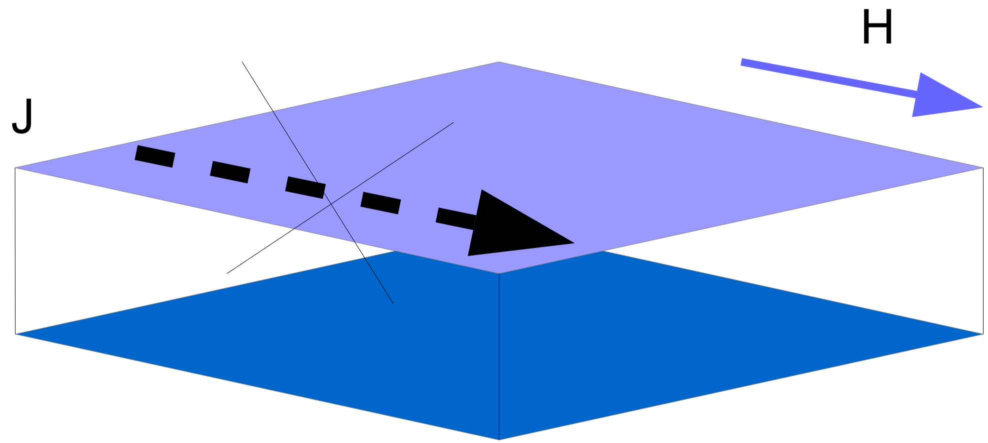
S. N. Valgushev, M. Puhr, and P. V. Buividovich, Chiral magnetic effect in finite-size samples of parity-breaking Weyl semimetals, arXiv:1512.01405.

P. V. Buividovich, M. Puhr, and S. N. Valgushev, Chiral magnetic conductivity in an interacting lattice model of parity-breaking Weyl semimetal, Phys. Rev. B **92**, 205122 (2015).

P. V. Buividovich, Spontaneous chiral symmetry breaking and the chiral magnetic effect for interacting Dirac fermions with chiral imbalance, Phys. Rev. D **90**, 125025 (2014).

P. V. Buividovich, Anomalous transport with overlap fermions, Nucl. Phys. **A925**, 218 (2014).

Chiral Magnetic Effect (CME) is the appearance of electric current in the direction of the external magnetic field in the presence of chiral chemical potential



Later the existance of the equilibrium static bulk CME was questioned.

Weyl semimetals M. Vazifeh and M. Franz, Electromagnetic Response of Weyl Semimetals, Phys. Rev. Lett. **111**, 027201 (2013).

Analysis based on the attempt to apply Bloch theorem

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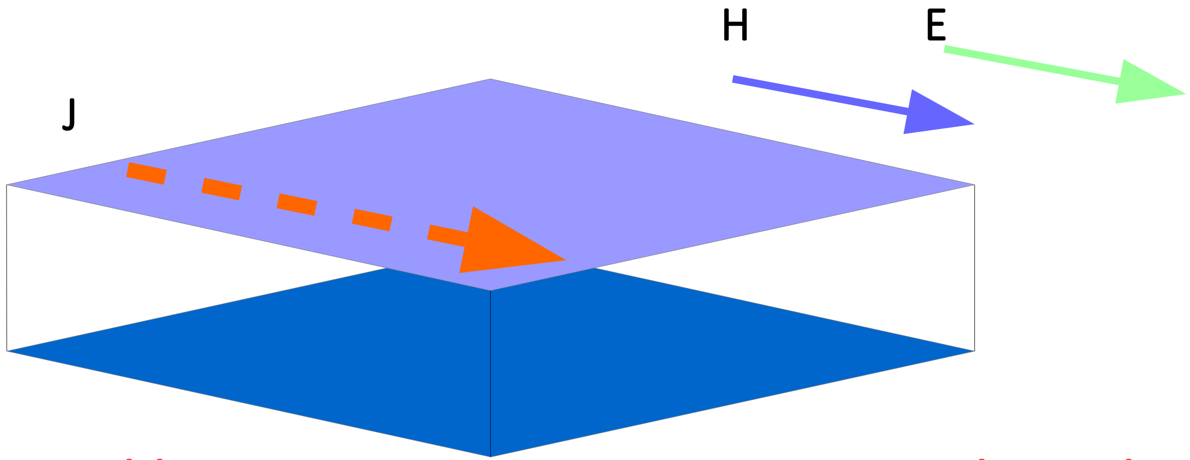
N. Yamamoto, Generalized Bloch theorem and chiral transport phenomena, Phys. Rev. D **92**, 085011 (2015).

different versions of CME



2) nonequilibrium CME in Dirac semimetals in the presence of both magnetic and electric fields

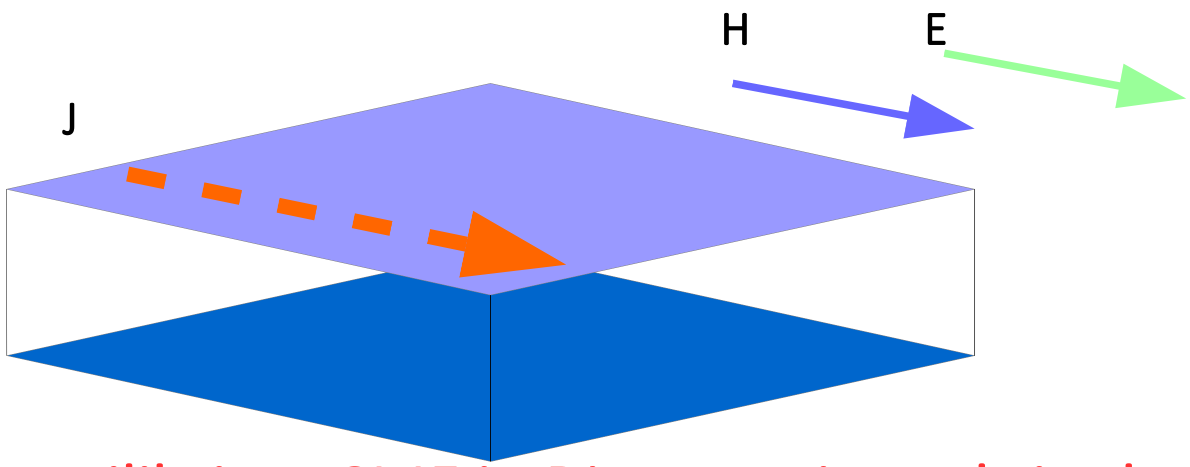
the chiral anomaly produces chiral Imbalance

this production requires energy taken from the job performed by the electic field.

This assumes existence of electric current j

JE = energy created while pumping the pairs from vacuum => J ~ B^2 E ?

different versions of CME



2) nonequilibrium CME in Dirac semimetals in the presence of emergent magnetic field (say, due to the dislocations)

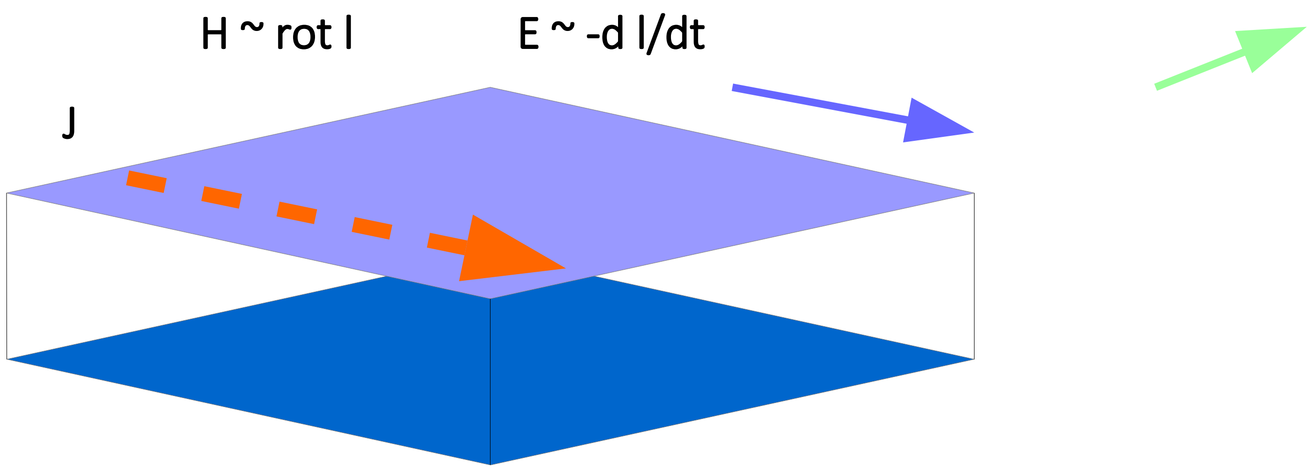
the chiral anomaly produces chiral Imbalance

this production requires energy taken from the job performed by the electic field.

This assumes existence of electric current j

JE = energy created while pumping the pairs from vacuum => J ~ ?

different versions of CME



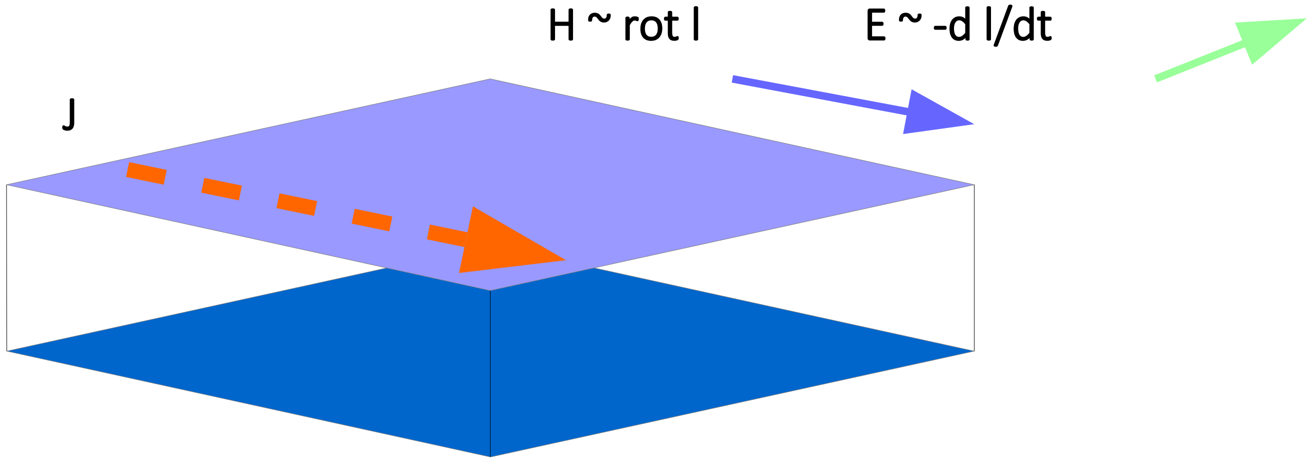
3) CME in He3-A, where mu_5 ~ l (v_n-v_s)

The applied technique for the calculation of the CME current does not work here because :

- the problem is not equilibrium

- the gauge field is emergent rather than real

different versions of CME



3) CME in He3-A, where mu_5 ~ l (v_n-v_s)

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- the problem is not equilibrium

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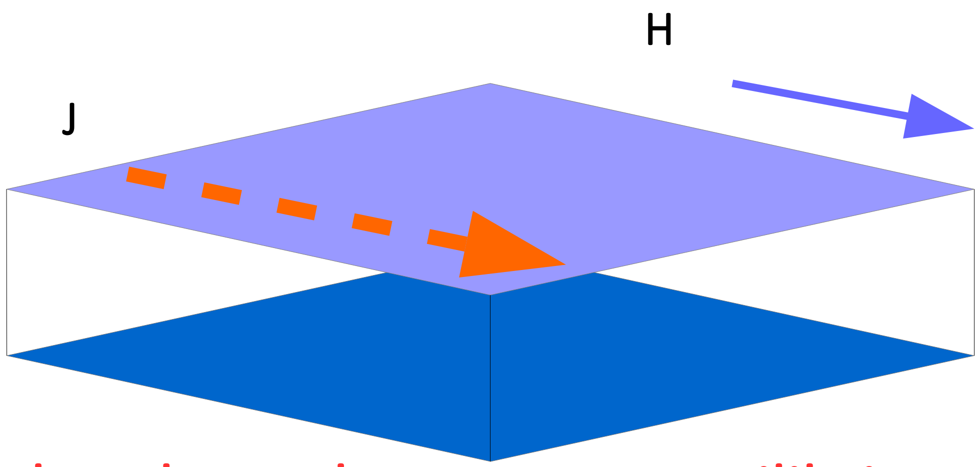
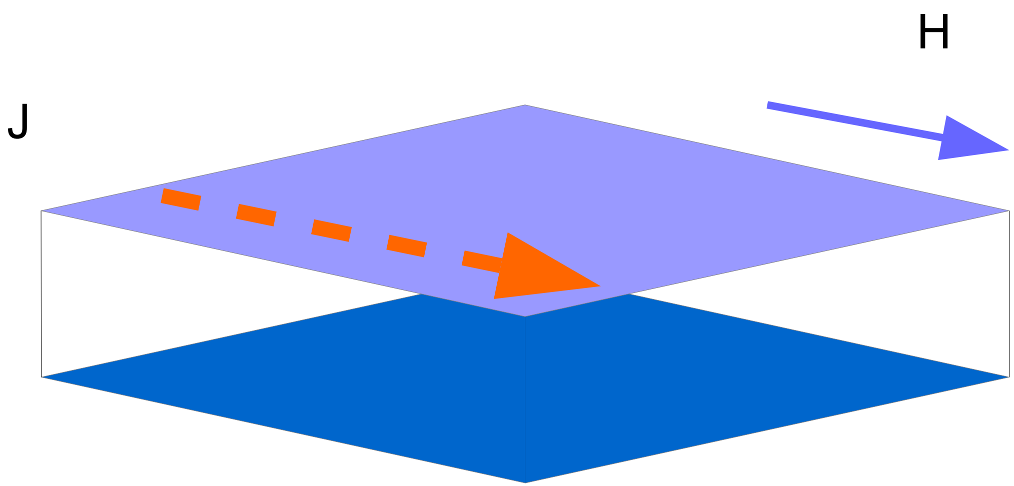
different forms of CME



4) Quark – gluon plasma : nonequilibrium CME contributions to the kinetic equations in the presence of the chiral imbalance?

Chiral imbalance that is described by chiral density rather than the chiral chemical potential ?

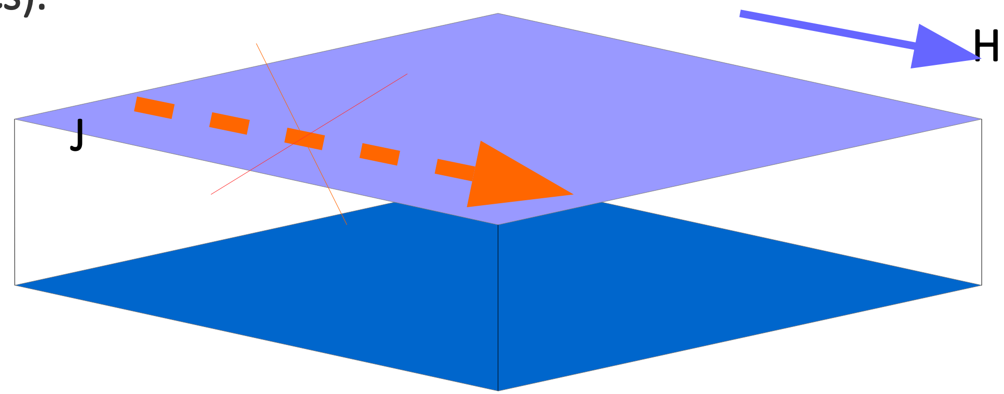
different forms of CME



4) QCD: contribution to electric conductivity in the presence of magnetic field

P. V. Buividovich, M. N. Chernodub, D. E. Kharzeev, T. Kalaydzhyan, E. V. Luschevskaya, and M. I. Polikarpov, Magnetic-Field-Induced Insulator-Conductor Transition in SU(2) Quenched Lattice Gauge Theory, Phys. Rev. Lett. **105**, 132001 (2010).

We considered lattice models with both massive and massless fermions that describe lattice regularized quantum field theory or the insulators and Dirac semimetals whose excitations are described by massive/massless Dirac action (in solid state physics).



J = M_4 /2pi^2 H   M_4 = 0 as long as mu_5 is nonzero

Chiral imbalance is described by the appearance of the chiral chemical potential

Green function (without external magnetic field) is:

$$\mathcal{G}(\mathbf{p}) = \Big(\sum_k \gamma^k g_k(\mathbf{p}) + i\gamma^4\gamma^5\mu_5 - im(\mathbf{p})\Big)^{-1}$$

There is no equilibrium static bulk CME

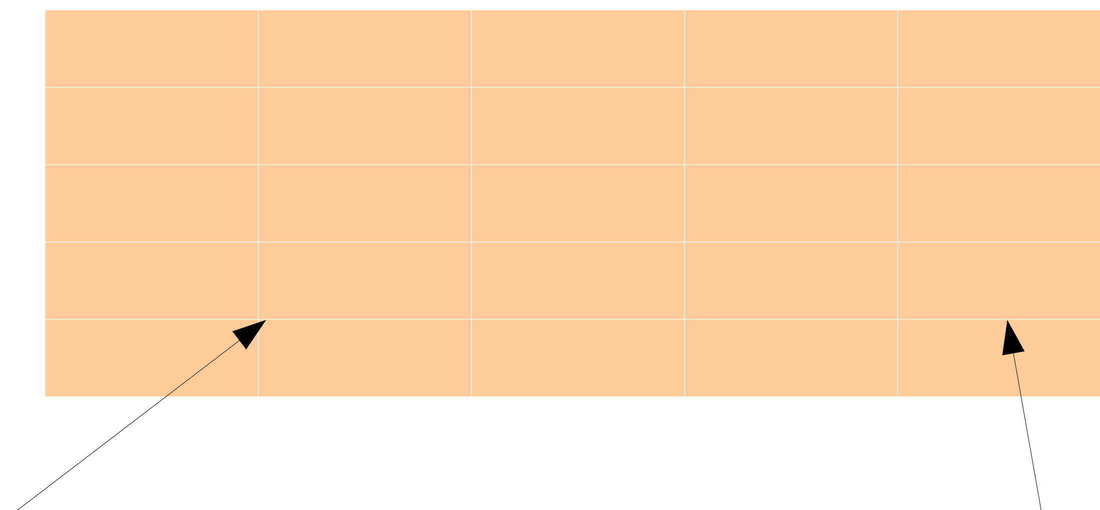
We work with the wide class of lattice models

example

Wilson fermions $Z = \int D\bar{\Psi} D\Psi \exp\Big(- \sum_{\mathbf{r}_n, \mathbf{r}_m} \bar{\Psi}(\mathbf{r}_m)(-i\mathcal{D}_{\mathbf{r}_n,\mathbf{r}_m})\Psi(\mathbf{r}_n)\Big)$

$$\mathcal{D}_{\mathbf{x},\mathbf{y}} = -\frac{1}{2}\sum_i [(1+\gamma^i)\delta_{\mathbf{x}+\mathbf{e}_i,\mathbf{y}} e^{iA_{\mathbf{x}+\mathbf{e}_i,\mathbf{y}}} + (1-\gamma^i)\delta_{\mathbf{x}-\mathbf{e}_i,\mathbf{y}} e^{iA_{\mathbf{x}-\mathbf{e}_i,\mathbf{y}}}] + (m^{(0)}+4)\delta_{\mathbf{x}\mathbf{y}}$$

The lattice:



Fermions are attached to sites (points).

Gauge field is attached to links that connect sites.

the model defined in momentum space:

$$Z = \int D\bar{\Psi} D\Psi \exp\Big(- \int_{\mathcal{M}} \frac{d^D \mathbf{p}}{|\mathcal{M}|} \bar{\Psi}(\mathbf{p}) \mathcal{G}^{-1}(\mathbf{p}) \Psi(\mathbf{p}) \Big)$$

coordinates are discrete ==> momentum space is compact
(electrons in solids and lattice regularized QFT)
In coordinate space

$$\Psi(\mathbf{r}) = \int_{\mathcal{M}} \frac{d^D \mathbf{p}}{|\mathcal{M}|} e^{i\mathbf{p}\mathbf{r}} \Psi(\mathbf{p})$$

$$Z = \int D\bar{\Psi} D\Psi \exp\Big(- \sum_{\mathbf{r}_n} \bar{\Psi}(\mathbf{r}_n) \Big[\mathcal{G}^{-1}(-i\partial_{\mathbf{r}}) \Psi(\mathbf{r}) \Big]_{\mathbf{r}=\mathbf{r}_n} \Big)$$

Example: Wilson fermions (=simple model of top.insulator)

$$\mathcal{G}(\mathbf{p}) = \Big(\sum_k \gamma^k g_k(\mathbf{p}) - i m(\mathbf{p}) \Big)^{-1}$$

$$g_k(\mathbf{p}) = \sin p_k, \quad m(\mathbf{p}) = m^{(0)} + \sum_{a=1,2,3,4} (1 - \cos p_a)$$

How to introduce the gauge field

Wilson fermions $Z = \int D\bar{\Psi} D\Psi \exp\Big(- \sum_{\mathbf{r}_n,\mathbf{r}_m} \bar{\Psi}(\mathbf{r}_m)(-i\mathcal{D}_{\mathbf{r}_n,\mathbf{r}_m})\Psi(\mathbf{r}_n)\Big)$

$$\mathcal{D}_{\mathbf{x},\mathbf{y}} = -\frac{1}{2}\sum_i [(1+\gamma^i)\delta_{\mathbf{x}+\mathbf{e}_i,\mathbf{y}} e^{iA_{\mathbf{x}+\mathbf{e}_i,\mathbf{y}}} + (1-\gamma^i)\delta_{\mathbf{x}-\mathbf{e}_i,\mathbf{y}} e^{iA_{\mathbf{x}-\mathbf{e}_i,\mathbf{y}}}] + (m^{(0)}+4)\delta_{\mathbf{xy}}$$

In momentum space:

$$\hat{\mathcal{Q}} = \mathcal{G}^{-1}(\mathbf{p} - \mathbf{A}(i\partial_{\mathbf{p}}))$$

$p_{i_1}...p_{i_n}$ ==> $\frac{1}{n!}\sum_{\text{permutations}}(\hat{p}_{i_1} - A_{i_1})...(\hat{p}_{i_n} - A_{i_n})$

$$Z = \int D\bar{\Psi} D\Psi \exp\Big(- \int_{\mathcal{M}} \frac{d^D\mathbf{p}}{|\mathcal{M}|} \bar{\Psi}(\mathbf{p})\hat{\mathcal{Q}}(i\partial_{\mathbf{p}},\mathbf{p})\Psi(\mathbf{p})\Big)$$

For Wilson fermions the equivalence is exact. For the other models it is up to the irrelevant terms ~a^2 x field strength

Gauge field appears as the pseudo — differential operator in momentum space.

Wigner transformation in coordinate space

Two point
Green function

$$G(\mathbf{r}_1,\mathbf{r}_2) = \frac{1}{Z}\int D\bar{\Psi}D\Psi\bar{\Psi}(\mathbf{r}_2)\Psi(\mathbf{r}_1) \times \exp\left(-\int d^D\mathbf{r}\bar{\Psi}(\mathbf{r})\hat{Q}(\mathbf{r},\hat{\mathbf{p}})\Psi(\mathbf{r})\right)$$

$$\hat{Q}(\mathbf{r}_1,-i\partial_{\mathbf{r}_1})G(\mathbf{r}_1,\mathbf{r}_2) = \delta^{(D)}(\mathbf{r}_1-\mathbf{r}_2)$$

Wigner transformation:

$$\tilde{G}(\mathbf{R},\mathbf{p}) = \int d^D\mathbf{r}e^{-i\mathbf{p}\mathbf{r}}G(\mathbf{R}+\mathbf{r}/2,\mathbf{R}-\mathbf{r}/2)$$

Weyl symbol of operator
=Wigner transform of matrix
element

$$Q(\mathbf{R},\mathbf{p}) = \int d^D\mathbf{x}d^D\mathbf{r}e^{-i\mathbf{p}\mathbf{r}}\delta(\mathbf{R}-\mathbf{r}/2-\mathbf{x}) \times \hat{Q}(\mathbf{x},-i\partial_{\mathbf{x}})\delta(\mathbf{R}+\mathbf{r}/2-\mathbf{x}).$$

F. A. Berezin and M. A. Shubin, in *Colloquia Mathematica Societatis Janos Bolyai* (North-Holland, Amsterdam, 1972), p. 21.

Robert G. Littlejohn, The semiclassical evolution of wave packets, Phys. Rep. **138**, 193 (1986).

Wigner transformation in coordinate space

$$G(\mathbf{r}_1,\mathbf{r}_2) = \frac{1}{Z}\int D\bar{\Psi}D\Psi\bar{\Psi}(\mathbf{r}_2)\Psi(\mathbf{r}_1)$$

Two point
Green function

$$\times \exp\left(-\int d^D\mathbf{r}\bar{\Psi}(\mathbf{r})\hat{Q}(\mathbf{r},\hat{\mathbf{p}})\Psi(\mathbf{r})\right)$$

$$\hat{Q}(\mathbf{r}_1,-i\partial_{\mathbf{r}_1})G(\mathbf{r}_1,\mathbf{r}_2) = \delta^{(D)}(\mathbf{r}_1-\mathbf{r}_2)$$

Wigner transformation:

$$\tilde{G}(\mathbf{R},\mathbf{p}) = \int d^D\mathbf{r}e^{-i\mathbf{p}\mathbf{r}}G(\mathbf{R}+\mathbf{r}/2,\mathbf{R}-\mathbf{r}/2)$$

Groenewold equation

$$\begin{aligned} 1 &= \mathcal{Q}(\mathbf{R},\mathbf{p}) * \tilde{G}(\mathbf{R},\mathbf{p}) \\ &\equiv \mathcal{Q}(\mathbf{R},\mathbf{p}) e^{\frac{i}{2}(\overleftarrow{\partial}_{\mathbf{R}}\overrightarrow{\partial}_{\mathbf{p}} - \overleftarrow{\partial}_{\mathbf{p}}\overrightarrow{\partial}_{\mathbf{R}})}\tilde{G}(\mathbf{R},\mathbf{p}) \end{aligned}$$

Weyl symbol of operator

If $\hat{\mathcal{Q}}(\mathbf{r},\hat{\mathbf{p}}) = \mathcal{G}^{-1}(\mathbf{p}-\mathbf{A}(i\partial_{\mathbf{p}}))$ ==> $\mathcal{Q}(\mathbf{r},\mathbf{p}) = \mathcal{G}^{-1}(\mathbf{p}-\mathbf{A}(\mathbf{r})) + O([\partial_i A_j]^2)$

For Wilson fermions the relation is exact

Wigner transformation in momentum space

Two point
Green function

$$G(\mathbf{p}_1,\mathbf{p}_2) = \frac{1}{Z}\int D\bar{\Psi}D\Psi\,\bar{\Psi}(\mathbf{p}_2)\Psi(\mathbf{p}_1)$$
$$\exp\Big(-\int\frac{d^D\mathbf{p}}{|\mathcal{M}|}\bar{\Psi}(\mathbf{p})\hat{Q}(i\partial_{\mathbf{p}},\mathbf{p})\Psi(\mathbf{p})\Big)$$

Wigner transformation:

$$\tilde{G}(\mathbf{R},\mathbf{p}) = \int\frac{d^D\mathbf{P}}{|\mathcal{M}|}e^{i\mathbf{P}\mathbf{R}}G(\mathbf{p}+\mathbf{P}/2,\mathbf{p}-\mathbf{P}/2)$$

In coordinate space:

$$\tilde{G}(\mathbf{R},\mathbf{p}) = \sum_{\mathbf{r}=\mathbf{r}_n}e^{-i\mathbf{p}\mathbf{r}}G(\mathbf{R}+\mathbf{r}/2,\mathbf{R}-\mathbf{r}/2)$$

$$G(\mathbf{r}_1,\mathbf{r}_2) = \frac{1}{Z}\int D\bar{\Psi}D\Psi\,\bar{\Psi}(\mathbf{r}_2)\Psi(\mathbf{r}_1)$$
$$\exp\Big(-\frac{1}{2}\sum_{\mathbf{r}_n}\Big[\bar{\Psi}(\mathbf{r}_n)\Big[\mathcal{G}^{-1}(-i\partial_{\mathbf{r}}$$
$$-\mathbf{A}(\mathbf{r}))\Psi(\mathbf{r})\Big]_{\mathbf{r}=\mathbf{r}_n}+(h.c.)\Big]\Big)$$

Wigner transformation in momentum space

Two point
Green function

$$G(\mathbf{p}_1,\mathbf{p}_2) = \frac{1}{Z}\int D\bar{\Psi}D\Psi\,\bar{\Psi}(\mathbf{p}_2)\Psi(\mathbf{p}_1)$$
$$\exp\Big(-\int\frac{d^D\mathbf{p}}{|\mathcal{M}|}\bar{\Psi}(\mathbf{p})\hat{Q}(i\partial_{\mathbf{p}},\mathbf{p})\Psi(\mathbf{p})\Big)$$

Wigner transformation:

$$\tilde{G}(\mathbf{R},\mathbf{p}) = \int\frac{d^D\mathbf{P}}{|\mathcal{M}|}e^{i\mathbf{P}\mathbf{R}}G(\mathbf{p}+\mathbf{P}/2,\mathbf{p}-\mathbf{P}/2)$$

$$1 = \mathcal{Q}(\mathbf{R},\mathbf{p}) * G(\mathbf{R},\mathbf{p})$$

Groenewold equation

$$\equiv \mathcal{Q}(\mathbf{R},\mathbf{p})e^{\frac{i}{2}(\overleftarrow{\partial}_{\mathbf{R}}\overrightarrow{\partial}_{\mathbf{p}}-\overleftarrow{\partial}_{\mathbf{p}}\overrightarrow{\partial}_{\mathbf{R}})}\tilde{G}(\mathbf{R},\mathbf{p})$$

Weyl symbol of operator

$$\mathcal{Q}(\mathbf{R},\mathbf{p}) = \int d^D\mathbf{K}d^D\mathbf{P}e^{i\mathbf{P}\mathbf{R}}\delta(\mathbf{p}-\mathbf{P}/2-\mathbf{K})$$
$$\times\hat{\mathcal{Q}}(i\partial_{\mathbf{K}},\mathbf{K})\delta(\mathbf{p}+\mathbf{P}/2-\mathbf{K}).$$

Wigner transform of
matrix element

$$\int d^D\mathbf{X}d^D\mathbf{Y}f(\mathbf{X},\mathbf{Y})\mathcal{Q}(-i\overleftarrow{\partial}_{\mathbf{Y}}+i\overrightarrow{\partial}_{\mathbf{X}},\mathbf{X}/2+\mathbf{Y}/2)h(\mathbf{X},\mathbf{Y})$$
$$= \int d^D\mathbf{X}d^D\mathbf{Y}f(\mathbf{X},\mathbf{Y})\hat{\mathcal{Q}}(i\partial_{\mathbf{X}}+i\partial_{\mathbf{Y}},\mathbf{X}/2+\mathbf{Y}/2)h(\mathbf{X},\mathbf{Y})$$

Wigner transformation in momentum space

Two point
Green function

$$G(\mathbf{p}_1,\mathbf{p}_2) = \frac{1}{Z}\int D\bar{\Psi}D\Psi\,\bar{\Psi}(\mathbf{p}_2)\Psi(\mathbf{p}_1)$$
$$\exp\Big(-\int\frac{d^D\mathbf{p}}{|\mathcal{M}|}\bar{\Psi}(\mathbf{p})\hat{Q}(i\partial_{\mathbf{p}},\mathbf{p})\Psi(\mathbf{p})\Big)$$

Wigner transformation:

$$\tilde{G}(\mathbf{R},\mathbf{p}) = \int\frac{d^D\mathbf{P}}{|\mathcal{M}|}e^{i\mathbf{P}\mathbf{R}}G(\mathbf{p}+\mathbf{P}/2,\mathbf{p}-\mathbf{P}/2)$$

Groenewold equation

$$1 = \mathcal{Q}(\mathbf{R},\mathbf{p})*\tilde{G}(\mathbf{R},\mathbf{p})$$
$$\equiv \mathcal{Q}(\mathbf{R},\mathbf{p})e^{\frac{i}{2}(\overleftarrow{\partial}_{\mathbf{R}}\overrightarrow{\partial}_{\mathbf{p}}-\overleftarrow{\partial}_{\mathbf{p}}\overrightarrow{\partial}_{\mathbf{R}})}\tilde{G}(\mathbf{R},\mathbf{p})$$

Weyl symbol of operator

If $\hat{\mathcal{Q}}(\mathbf{r},\hat{\mathbf{p}}) = \mathcal{G}^{-1}(\mathbf{p}-\mathbf{A}(i\partial_{\mathbf{p}}))$ ==> $\mathcal{Q}(\mathbf{r},\mathbf{p}) = \mathcal{G}^{-1}(\mathbf{p}-\mathbf{A}(\mathbf{r}))+O([\partial_i A_j]^2)$

For Wilson fermions the relation is exact

Solution of Groenewold equation

Weak
external gauge field

$$\begin{aligned} 1 &= \mathcal{Q}(\mathbf{R},\mathbf{p}) * \tilde{G}(\mathbf{R},\mathbf{p}) \\ &\equiv \mathcal{Q}(\mathbf{R},\mathbf{p}) e^{\frac{i}{2}(\overleftarrow{\partial}_{\mathbf{R}} \overrightarrow{\partial}_{\mathbf{p}} - \overleftarrow{\partial}_{\mathbf{p}} \overrightarrow{\partial}_{\mathbf{R}})} \tilde{G}(\mathbf{R},\mathbf{p}) \end{aligned}$$

$$\begin{aligned} \tilde{G}(\mathbf{R},\mathbf{p}) &= \tilde{G}^{(0)}(\mathbf{R},\mathbf{p}) + \tilde{G}^{(1)}(\mathbf{R},\mathbf{p}) + \ldots \\ \tilde{G}^{(1)} &= -\frac{i}{2} \tilde{G}^{(0)} \frac{\partial \left[\tilde{G}^{(0)}\right]^{-1}}{\partial p_i} \tilde{G}^{(0)} \frac{\partial \left[\tilde{G}^{(0)}\right]^{-1}}{\partial p_j} \tilde{G}^{(0)} A_{ij}(\mathbf{R}) \end{aligned}$$

with

$$\tilde{G}^{(0)}(\mathbf{R},\mathbf{p}) = \mathcal{G}(\mathbf{p} - \mathbf{A}(\mathbf{R}))$$

Response of electric current to the gauge field

$$j^k(\mathbf{R}) = \int_{\mathcal{M}} \frac{d^D\mathbf{p}}{|\mathcal{V}||\mathcal{M}|} \mathrm{Tr}\, \tilde{G}(\mathbf{R},\mathbf{p}) \frac{\partial}{\partial p_k} \left[\tilde{G}^{(0)}(\mathbf{R},\mathbf{p})\right]^{-1}$$

$(2\pi)^D$

Response of current to the gauge field strength

$$\mathbf{A} \to \mathbf{A} + \delta\mathbf{A}$$

$$\delta \log Z = -\frac{1}{Z}\int D\bar{\Psi} D\Psi \exp\Big(- \int_{\mathcal{M}} \frac{d^D\mathbf{p}}{|\mathcal{M}|} \bar{\Psi}(\mathbf{p}) \hat{\mathcal{Q}}(i\partial_{\mathbf{p}}, \mathbf{p}) \Psi(\mathbf{p})\Big) \int_{\mathcal{M}} \frac{d^D\mathbf{p}}{|\mathcal{M}|} \bar{\Psi}(\mathbf{p}) \Big[\delta \hat{\mathcal{Q}}(i\partial_{\mathbf{p}}, \mathbf{p})\Big] \Psi(\mathbf{p})$$

$$= -\int_{\mathcal{M}} \frac{d^D\mathbf{p}}{|\mathcal{M}|} \mathrm{Tr}\Big[\delta \hat{\mathcal{Q}}(i\partial_{\mathbf{p}_1}, \mathbf{p}_1)\Big] G(\mathbf{p}_1, \mathbf{p}_2)\Big|_{\mathbf{p}_1=\mathbf{p}_2=\mathbf{p}}$$

$$= -\sum_{\mathbf{R}=\mathbf{R}_n} \int_{\mathcal{M}} \frac{d^D\mathbf{p}}{|\mathcal{M}|} \mathrm{Tr}\Big[\delta \hat{\mathcal{Q}}(i\partial_{\mathbf{P}} + i\partial_{\mathbf{p}}/2, \mathbf{p} + \mathbf{P}/2)\Big] e^{-i\mathbf{P}\mathbf{R}} \tilde{G}(\mathbf{R}, \mathbf{p})\Big|_{\mathbf{P}=0}$$

$$\delta \log Z = -\sum_{\mathbf{R}=\mathbf{R}_n} \int_{\mathcal{M}} \frac{d^D\mathbf{p}}{|\mathcal{M}|} \mathrm{Tr}\Big[\delta \mathcal{Q}(i\overrightarrow{\partial}_{\mathbf{P}} - i\overleftarrow{\partial}_{\mathbf{p}}/2, \mathbf{p} + \mathbf{P}/2)\Big] e^{-i\mathbf{P}\mathbf{R}} \tilde{G}(\mathbf{R}, \mathbf{p})\Big|_{\mathbf{P}=0}$$

$$= -\sum_{\mathbf{R}=\mathbf{R}_n} \int_{\mathcal{M}} \frac{d^D\mathbf{p}}{|\mathcal{M}|} \mathrm{Tr}\Big[\delta \mathcal{Q}(\mathbf{R}, \mathbf{p} + \mathbf{P}/2)\Big] e^{-i\mathbf{P}\mathbf{R}} \tilde{G}(\mathbf{R}, \mathbf{p})\Big|_{\mathbf{P}=0}$$

Response of electric current to the gauge field

$$j^k(\mathbf{R}) = \int_{\mathcal{M}} \frac{d^D\mathbf{p}}{|\mathcal{V}||\mathcal{M}|} \mathrm{Tr}\, \tilde{G}(\mathbf{R}, \mathbf{p}) \frac{\partial}{\partial p_k} \Big[\tilde{G}^{(0)}(\mathbf{R}, \mathbf{p})\Big]^{-1}$$

$$\delta \log Z = \sum_{\mathbf{R}=\mathbf{R}_n} j^k(\mathbf{R}) \delta A_k(\mathbf{R}) |\mathcal{V}| \qquad (2\pi)^D$$

Response of electric current to the gauge field

$$j^k(\mathbf{R}) = j^{(0)k}(\mathbf{R}) + j^{(1)k}(\mathbf{R}) + \ldots$$

$$j^{(0)k}(\mathbf{R}) = \int \frac{d^D\mathbf{p}}{(2\pi)^D} \,\mathrm{Tr}\, \tilde{G}^{(0)}(\mathbf{R},\mathbf{p}) \frac{\partial \left[\tilde{G}^{(0)}(\mathbf{R},\mathbf{p})\right]^{-1}}{\partial p_k}$$

with

$$j^{(1)k}(\mathbf{R}) = \frac{1}{4\pi^2}\epsilon^{ijkl}\mathcal{M}_l A_{ij}(\mathbf{R}),$$

$$\mathcal{M}_l = \int \mathrm{Tr}\,\nu_l\, d^4p$$

3+1 D

$$\nu_l = -\frac{i}{3!\,8\pi^2}\,\epsilon_{ijkl}\left[\mathcal{G}\frac{\partial\mathcal{G}^{-1}}{\partial p_i}\frac{\partial\mathcal{G}}{\partial p_j}\frac{\partial\mathcal{G}^{-1}}{\partial p_k}\right]$$

To have well — defined expressions we need:

1) Ultraviolet regularization **MASSIVE**

2) Infrared regularization **LATTCE FERMIONS**

Response of electric current to the gauge field

$$
\begin{aligned}
j^k(\mathbf{R}) &= j^{(0)k}(\mathbf{R}) + j^{(1)k}(\mathbf{R}) + \ldots \\
j^{(0)k}(\mathbf{R}) &= \int \frac{d^D\mathbf{p}}{(2\pi)^D}\,\mathrm{Tr}\,\tilde{G}^{(0)}(\mathbf{R},\mathbf{p})\frac{\partial\left[\tilde{G}^{(0)}(\mathbf{R},\mathbf{p})\right]^{-1}}{\partial p_k}
\end{aligned}
$$

with

2+1 D

$$
\begin{aligned}
j^{(1)k}(\mathbf{R}) &= \frac{1}{4\pi}\epsilon^{ijk}\mathcal{M}A_{ij}(\mathbf{R}), \quad \mathcal{M} = \int \mathrm{Tr}\,\nu\, d^3p \\
\nu &= -\frac{i}{3!\,4\pi^2}\,\epsilon_{ijk}\left[\tilde{G}^{(0)}(\mathbf{R},\mathbf{p})\frac{\partial\left[\tilde{G}^{(0)}(\mathbf{R},\mathbf{p})\right]^{-1}}{\partial p_i}\frac{\partial\left[\tilde{G}^{(0)}(\mathbf{R},\mathbf{p})\right]}{\partial p_j}\frac{\partial\left[\tilde{G}^{(0)}(\mathbf{R},\mathbf{p})\right]^{-1}}{\partial p_k}\right]
\end{aligned}
$$

2+1 D Anomalous Quantum Hall effect 2D top. insulator

We reproduce

$$j^k_{Hall} = \frac{1}{2\pi}\tilde{\mathcal{N}}_3\,\epsilon^{ki}E_i$$

$$\mathbf{E} = (E_1, E_2) \text{ as } A_{3k} = -iE_k$$

G.E.Volovik,
JETP 67 (1988), 1804 — 1811

$$\tilde{\mathcal{N}}_3 = -\frac{1}{24\pi^2}\mathrm{Tr}\int \mathcal{G}^{-1}d\mathcal{G}\wedge d\mathcal{G}^{-1}\wedge d\mathcal{G}$$

In the particular case of the non — interacting system it is reduced to

$$\mathcal{G}^{-1} = i\omega - \hat{H}$$

$$\tilde{\mathcal{N}}_3 = \frac{\epsilon^{ij}}{4\pi}\sum_{k:\mathcal{E}_k<0}\int d^2p\,\mathcal{F}_{ij}$$

Berry curvature $\mathcal{F}_{ij} = \partial_i\mathcal{A}_j - \partial_j\mathcal{A}_i$ $\mathcal{A}_j = i\langle k,\vec{p}|\partial_j|k,\vec{p}\rangle$

D. J. Thouless, M. Kohmoto, M. P. Nightingale, M. den Nijs,
Phys. Rev. Lett. 49, 405 (1982)

2+1 D Anomalous Quantum Hall effect

Bulk — boundary correspondence

$$j^k_{Hall} = \frac{1}{2\pi}\tilde{\mathcal{N}}_3\,\epsilon^{ki}E_i$$

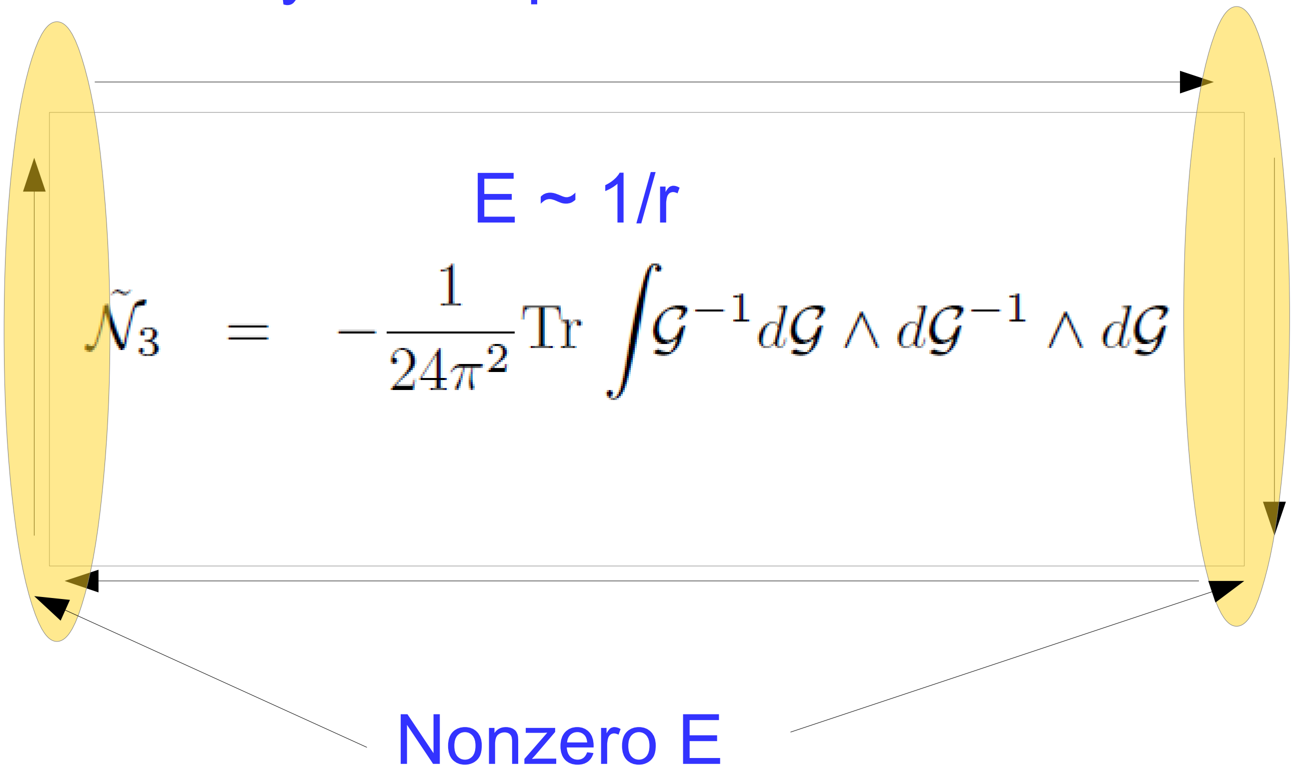
$$\tilde{\mathcal{N}}_3 \quad = \quad -\frac{1}{24\pi^2}\mathrm{Tr}\,\int \mathcal{G}^{-1}d\mathcal{G}\wedge d\mathcal{G}^{-1}\wedge d\mathcal{G}$$

The number of boundary gapless fermions = N_3 (G.E. Volovik, «The Universe in a Helium Droplet», Clarendon Press, Oxford (2003))

2+1 D Anomalous Quantum Hall effect

Bulk — boundary correspondence

$$j^k_{Hall} = \frac{1}{2\pi} \tilde{\mathcal{N}}_3 \, \epsilon^{ki} E_i$$



$$\tilde{\mathcal{N}}_3 = -\frac{1}{24\pi^2} \mathrm{Tr} \int \mathcal{G}^{-1} d\mathcal{G} \wedge d\mathcal{G}^{-1} \wedge d\mathcal{G}$$

The total current is given by N_3/2pi x V = J_left — J_right and is carried by the boundary gapless fermions

(we use relativistic units)

2+1 D Anomalous Quantum Hall effect

Bulk — boundary correspondence

$$j^k_{Hall} = \frac{1}{2\pi} \tilde{\mathcal{N}}_3 \, \epsilon^{ki} E_i$$

E = const

U=-V/2

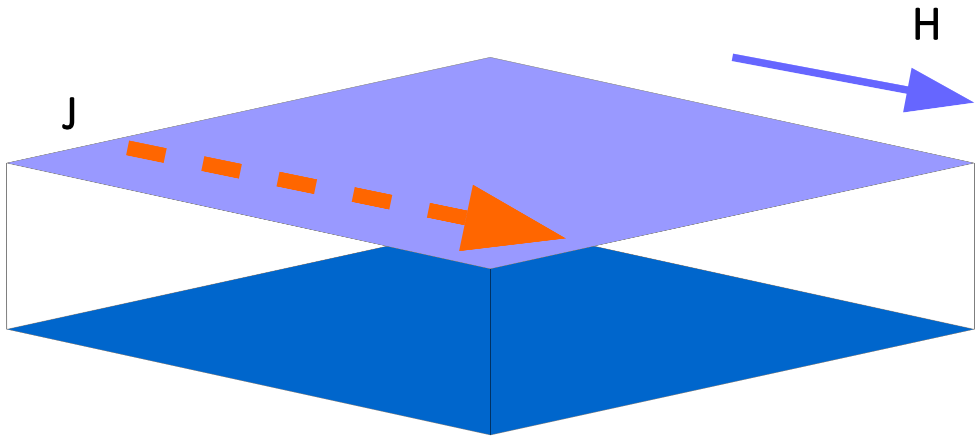
$$\tilde{\mathcal{N}}_3 = -\frac{1}{24\pi^2} \mathrm{Tr} \int \mathcal{G}^{-1} d\mathcal{G} \wedge d\mathcal{G}^{-1} \wedge d\mathcal{G}$$

U=V/2

E = V/L

In the ideal system the total current is given by N_3/2pi x V and is carried by the bulk.
The self — check: Loughlin geometry (opposite sides are indentified, and there are no boundaries that may carry the current).

Chiral Magnetic Effect (CME) is the appearance of electric current in the direction of the external magnetic field in the presence of chiral chemical potential

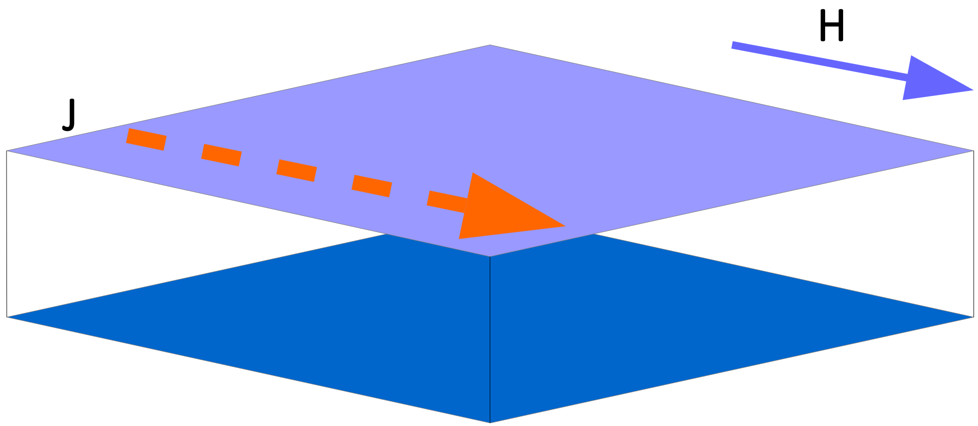


J = M /2pi^2 H   M = mu_5 ?

Pre – history: the existence of chiral magnetic effect was predicted 30 years ago by Vilenkin. This proposition was later repeated for quark-gluon plasma and for Dirac semimetals by D.Kharzeev. (The Dirac semimetals are the materials with emergent relativistic invariance). It was proposed, that magnetic field produces electric current directed along the magnetic field and proportional to it.

Later the very existance of this effect was questioned. It was demonstrated using numerical simulations, for example, by P. Buividovich (Regensburg U). that in certain lattice regularizations the simplest and the most commonly accepted version of this effect is absent. In condensed matter theory the existence of this effect was criticised, for example, by N.Yamamoto (his analysis was based on the attempt to apply Bloch theorem). In a certain model of solids the same conclusion was achieved by Vasifeh and Franz

We start from the lattice model with massive fermions that describes lattice regularized quantum field theory or the insulators whose excitations are described by massive Dirac action (in solid state physics).



J = M /2pi^2 H   M = mu_5 ?

Chiral imbalance is described by the appearance of the chiral chemical potential

Green function (without external magnetic field) is:

$$\mathcal{G}(\mathbf{p}) = \Big(\sum_k \gamma^k g_k(\mathbf{p}) + i\gamma^4\gamma^5\mu_5 - im(\mathbf{p})\Big)^{-1}$$

Example : Wilson fermions

$$g_k(\mathbf{p}) = \sin p_k, \quad m(\mathbf{p}) = m^{(0)} + \sum_{a=1,2,3,4} (1 - \cos p_a)$$

3+1 D Chiral Magnetic Effect 3D Dirac insulator

In lattice models
we obtain for the first time
$\mathcal{M}_4$ is responsible for the CME

$$j^{(1)k}(\mathbf{R}) = \frac{1}{4\pi^2}\epsilon^{ijkl}\mathcal{M}_l A_{ij}(\mathbf{R})$$

$$\mathcal{M}_l = \int \mathrm{Tr}\, \nu_l \, d^4p$$

In continuous models
this follows trivially
from Feinman diagrams
4x4 Green function

$$\nu_l = -\frac{i}{3!\, 8\pi^2}\,\epsilon_{ijkl}\left[\mathcal{G}\frac{\partial \mathcal{G}^{-1}}{\partial p_i}\frac{\partial \mathcal{G}}{\partial p_j}\frac{\partial \mathcal{G}^{-1}}{\partial p_k}\right]$$

$$\mathcal{G}(\mathbf{p}) = \Big(\sum_k \gamma^k g_k(\mathbf{p}) + i\gamma^4\gamma^5\mu_5 - im(\mathbf{p})\Big)^{-1}$$

Let us assume first, that without chiral chemical potential the fermions are gapped. Poles of the Green function may appear for the nonzero mu_5 if

$$g_4^2(\mathbf{p}) + \Big(\mu_5 \pm \sqrt{g_1^2(\mathbf{p}) + g_2^2(\mathbf{p}) + g_3^2(\mathbf{p})}\Big)^2 + m^2(\mathbf{p}) = 0$$

3+1 D Chiral Magnetic Effect 3D Dirac insulator

conventional case

Example:
Wilson fermions

$$\mathcal{G}(\mathbf{p}) = \Big(\sum_k \gamma^k g_k(\mathbf{p}) + i\gamma^4\gamma^5\mu_5 - im(\mathbf{p})\Big)^{-1}$$

With $m^{(0)} > 0$ $\quad g_k(\mathbf{p}) = \sin p_k, \quad m(\mathbf{p}) = m^{(0)} + \sum_{a=1,2,3,4} (1 - \cos p_a)$

Function m(p) never equals to zero ==> nonzero mu_5 cannot cause the poles of G. (The same is if in general case g_4(p) and m(p) do not vanish simultaneousely.)

M_4 is top. Invariant =>
=> we may calculate it
for mu_5 = 0.

$$\mathcal{M}_4 = -\frac{i}{2}\int dp^4 \tilde{\mathcal{N}}_3(p^4),$$

$$\tilde{\mathcal{N}}_3(p^4) = \frac{1}{24\pi^2}\epsilon_{ijk4}\mathrm{Tr}\int_\Omega d^3p\Big(\mathcal{G}\partial^i\mathcal{G}^{-1}\Big)\Big(\mathcal{G}\partial^j\mathcal{G}^{-1}\Big)\Big(\mathcal{G}\partial^k\mathcal{G}^{-1}\Big)$$

3+1 D Chiral Magnetic Effect 3D Dirac insulator

conventional case

$$\mathcal{G}(\mathbf{p}) = \Big(\sum_k \gamma^k g_k(\mathbf{p}) + i\gamma^4\gamma^5\mu_5 - im(\mathbf{p})\Big)^{-1}$$

Introduce

$$\Gamma^k = i\gamma^5\gamma^k \text{ for } k = 1, 2, 3, 4, \text{ and } \Gamma^5 = \gamma^5$$

Calculate

$$\tilde{\mathcal{N}}_3(p^4) = \frac{1}{24\pi^2}\epsilon_{ijk4}\,\mathrm{Tr}\,\Gamma^a\Gamma^b\Gamma^c\Gamma^d \int_\Omega d^3p \frac{g_a}{g^2}\partial^i g_b \partial^j\Big(\frac{g_c}{g^2}\Big)\partial^k g_d$$

$$g = \sqrt{\sum_{k=1,2,3,4,5} g_k^2}$$

$$= \frac{1}{6\pi^2}\epsilon_{ijk4}(\delta^{ab}\delta^{cd} - \delta^{ac}\delta^{bd} + \delta^{ad}\delta^{bc}) \int_\Omega d^3p \frac{g_a\partial^i g_b\Big(\partial^j g_c - g_c\partial^j \log g^2\Big)\partial^k g_d}{g^4}$$

Then deform smoothly G to the form with nonzero mu_5

$$= \frac{1}{6\pi^2}\epsilon_{ijk4}(\delta^{ab}\delta^{cd} + \delta^{ad}\delta^{bc}) \int_\Omega d^3p \frac{g_a\partial^i g_b\partial^j g_c\partial^k g_d}{g^4} = 0$$

3+1 D Chiral Magnetic Effect 3D Dirac insulator

conventional case N_5=0 *(M.A.Zubkov and G.E.Volovik.Nucl.Phys.B 860, 295 (2012))*

Example:
Wilson fermions

$$\mathcal{G}(\mathbf{p}) = \Big(\sum_k \gamma^k g_k(\mathbf{p}) + i\gamma^4\gamma^5\mu_5 - im(\mathbf{p})\Big)^{-1}$$

With $m^{(0)} > 0$ $\quad g_k(\mathbf{p}) = \sin p_k, \quad m(\mathbf{p}) = m^{(0)} + \sum_{a=1,2,3,4}(1-\cos p_a)$

Function m(p) never equals to zero ==> nonzero mu_5 cannot cause the poles of G. (The same is if in general case g_4(p) and m(p) do not vanish simultaneousely.)

M_4 = 0 we may calculate it $\mathcal{M}_4 = -\frac{i}{2}\int dp^4 \tilde{\mathcal{N}}_3(p^4),$
for mu_5 = 0, and then
for nonzero mu_5
its value is the same

$$\tilde{\mathcal{N}}_3(p^4) = \frac{1}{24\pi^2}\epsilon_{ijk4}\mathrm{Tr}\int_\Omega d^3p\Big(\mathcal{G}\partial^i\mathcal{G}^{-1}\Big)\Big(\mathcal{G}\partial^j\mathcal{G}^{-1}\Big)\Big(\mathcal{G}\partial^k\mathcal{G}^{-1}\Big)$$

3+1 D Chiral Magnetic Effect N_5=-2 (massless fermions appear at the phase transition between the two phases with different N_5)

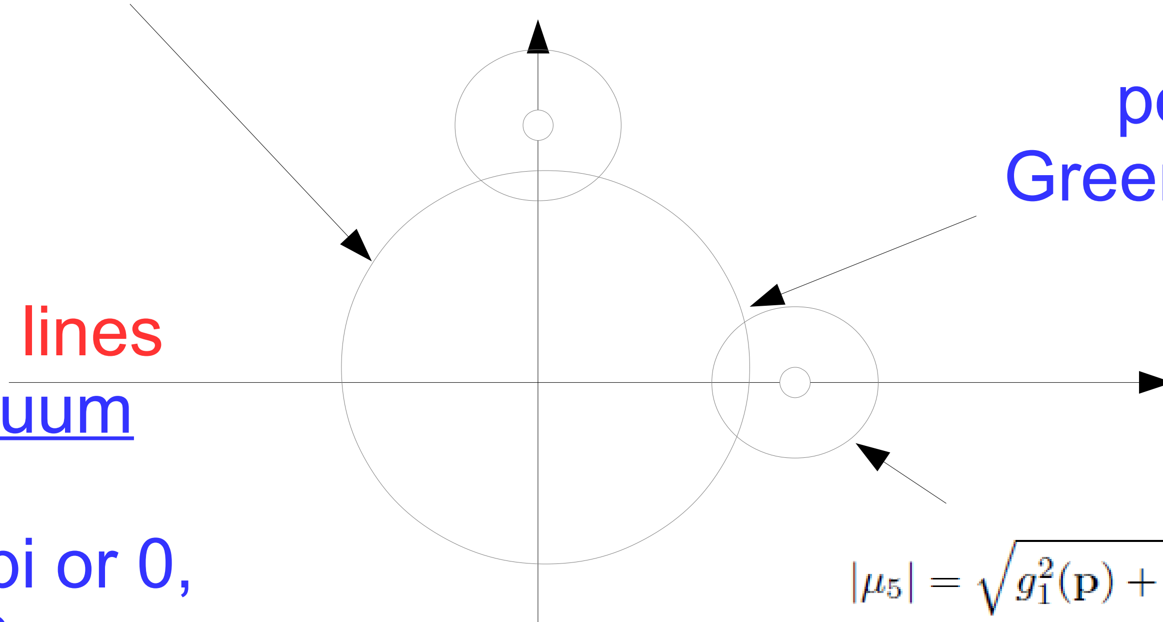
$$\mathcal{G}(\mathbf{p}) = \Big(\sum_k \gamma^k g_k(\mathbf{p}) + i\gamma^4\gamma^5\mu_5 - im(\mathbf{p})\Big)^{-1}$$

marginal example Wilson fermions with $m^{(0)} \in (-2, 0)$

$$g_k(\mathbf{p}) = \sin p_k, \quad m(\mathbf{p}) = m^{(0)} + \sum_{a=1,2,3,4} (1 - \cos p_a)$$

the zeros of m(p) form the curves (p_3=p_4=0)

pole of the Green function



$$|\mu_5| = \sqrt{g_1^2(\mathbf{p}) + g_2^2(\mathbf{p}) + g_3^2(\mathbf{p})}$$

closed Fermi lines

Marginal vacuum

If p_4 is not pi or 0, N_3(p_4) = 0

$$g_4^2(\mathbf{p}) + \Big(\mu_5 \pm \sqrt{g_1^2(\mathbf{p}) + g_2^2(\mathbf{p}) + g_3^2(\mathbf{p})}\Big)^2 + m^2(\mathbf{p}) = 0$$

3+1 D Chiral Magnetic Effect 3D Dirac insulator

marginal example with
Wilson fermions

$$\mathcal{M}_4 = -\frac{i}{2}\int dp^4 \tilde{\mathcal{N}}_3(p^4),$$

$$\tilde{\mathcal{N}}_3(p^4) = \frac{1}{24\pi^2}\epsilon_{ijk4}\mathrm{Tr}\int_\Omega d^3p\Big(\mathcal{G}\partial^i\mathcal{G}^{-1}\Big)\Big(\mathcal{G}\partial^j\mathcal{G}^{-1}\Big)\Big(\mathcal{G}\partial^k\mathcal{G}^{-1}\Big)$$

If we regularize the
Integral as

$$\int = \lim_{\epsilon\to 0}\Big(\int_{-\infty}^{-\epsilon} + \int_{\epsilon}^{\infty}\Big)$$

Then M_4 → 0 because N_3(p) = 0 for p that is not zero or pi

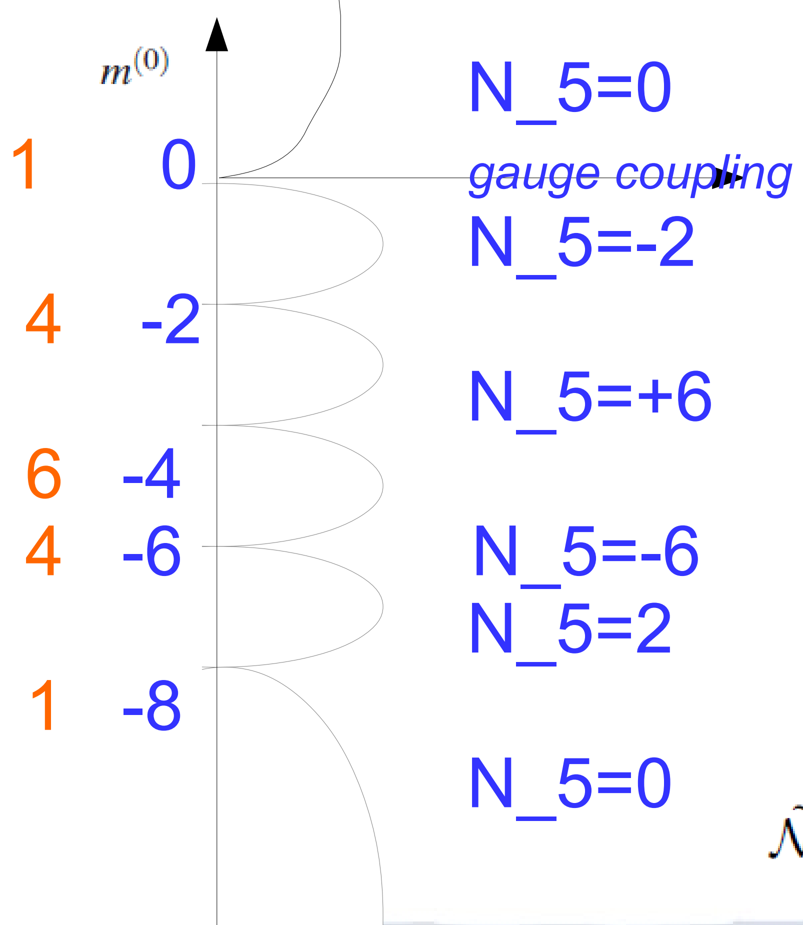
At finite temperature we use the sum over the Matsubara frequencies $\omega_n = T\pi(2n+1) \neq 0$
And the answer is the same

3D Dirac semimetal = lattice regularization of the QFT with Massless fermions *massless fermions appear at the phase transition between the two phases with different N_5*

Example: Wilson fermions

Phase diagram

$$g_k(\mathbf{p}) = \sin p_k, \qquad m(\mathbf{p}) = m^{(0)} + \sum_{a=1,2,3,4} (1 - \cos p_a)$$



$$\mathcal{G}(\mathbf{p})|_{\mu_5=0} = \Big(\sum_k \gamma^k g_k(\mathbf{p}) - i m(\mathbf{p})\Big)^{-1}$$

$$g_5[p] = m[p] \qquad \hat{g} = \frac{g}{\sqrt{g^a g^a}}$$

$$\tilde{\mathcal{N}}_5 = \frac{3}{4\pi^2 4!} \epsilon_{abcde} \int \hat{g}^a \, d\hat{g}^b \wedge d\hat{g}^c \wedge d\hat{g}^d \wedge d\hat{g}^e$$

3D Dirac semimetal = lattice regularization of the QFT with Massless fermions *massless fermions appear at the phase transition between the two phases with different N_5*

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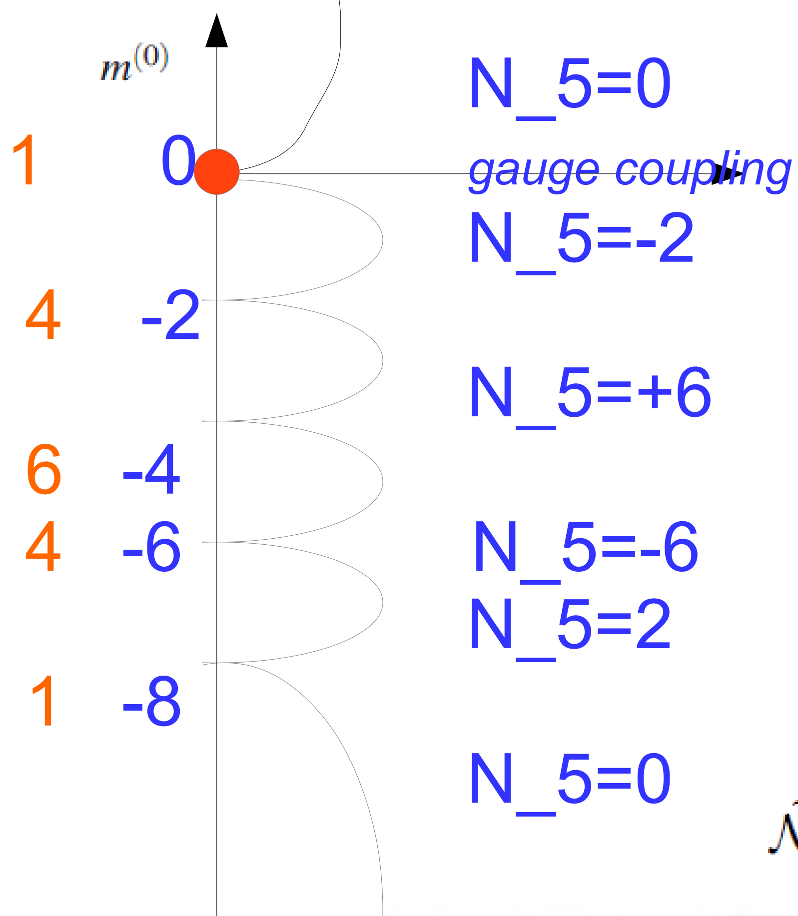

$$\mathcal{G}(\mathbf{p})|_{\mu_5=0} = \left(\sum_k \gamma^k g_k(\mathbf{p}) - i m(\mathbf{p})\right)^{-1}$$

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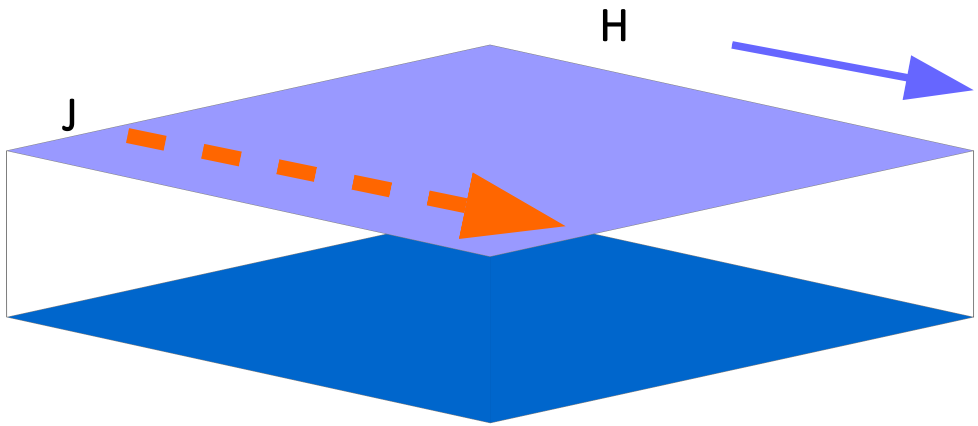
$$\tilde{\mathcal{N}}_5 = \frac{3}{4\pi^2 4!}\epsilon_{abcde} \int \hat{g}^a \, d\hat{g}^b \wedge d\hat{g}^c \wedge d\hat{g}^d \wedge d\hat{g}^e$$

We deal with the lattice model with massless fermion that describes lattice regularized quantum field theory or the Dirac semimetal (whose excitations are described by the massless Dirac action.



J = M /2pi^2 H   M = mu_5 ?

$$\mathcal{G}(\mathbf{p}) = \Big(\sum_k \gamma^k g_k(\mathbf{p}) + i\gamma^4\gamma^5\mu_5 - im(\mathbf{p})\Big)^{-1}$$

Chiral imbalance

Green function

$$j^{(1)k}(\mathbf{R}) = \frac{1}{4\pi^2}\epsilon^{ijkl}\mathcal{M}_l A_{ij}(\mathbf{R})$$

We are to calculate M_4

$$\mathcal{M}_4 = -\frac{i}{2}\int dp^4 \tilde{\mathcal{N}}_3(p^4),$$

$$\tilde{\mathcal{N}}_3(p^4) = \frac{1}{24\pi^2}\epsilon_{ijk4}\mathrm{Tr}\int_\Omega d^3p\Big(\mathcal{G}\partial^i\mathcal{G}^{-1}\Big)\Big(\mathcal{G}\partial^j\mathcal{G}^{-1}\Big)\Big(\mathcal{G}\partial^k\mathcal{G}^{-1}\Big)$$

3+1 D Chiral Magnetic Effect 3D Dirac semimetal

non - marginal
example:
Wilson fermions

$$\mathcal{G}(\mathbf{p}) = \Big(\sum_k \gamma^k g_k(\mathbf{p}) + i\gamma^4\gamma^5\mu_5 - im(\mathbf{p})\Big)^{-1}$$

With $m^{(0)} = 0 \quad g_k(\mathbf{p}) = \sin p_k, \quad m(\mathbf{p}) = m^{(0)} + \sum_{a=1,2,3,4} (1 - \cos p_a)$

For nonzero mu_5 there are no solutions of

$$g_4^2(\mathbf{p}) + \Big(\mu_5 \pm \sqrt{g_1^2(\mathbf{p}) + g_2^2(\mathbf{p}) + g_3^2(\mathbf{p})}\Big)^2 + m^2(\mathbf{p}) = 0$$

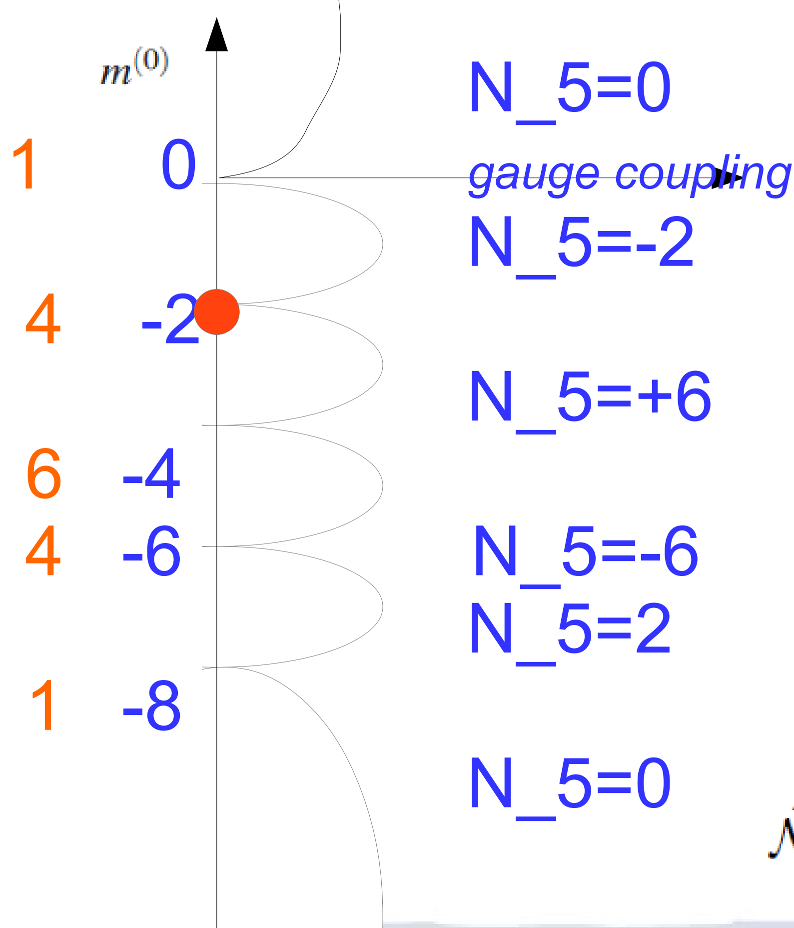
Therefore, for nonzero mu_5 there are no poles of G.
We may smoothly make m^{(0)} positive, and after that smoothly bring mu_5 to zero. This transformation does not encounter the poles of G.
This is the proof that M_4 = 0 for Dirac semimetals with nonzero chiral chemical potential.
The same refers to the lattice discretization of quantum field theory.

3D Dirac semimetal = lattice regularization of the QFT with Massless fermions *massless fermions appear at the phase transition between the two phases with different N_5*

Example: Wilson fermions

Phase diagram

$$g_k(\mathbf{p}) = \sin p_k, \quad m(\mathbf{p}) = m^{(0)} + \sum_{a=1,2,3,4} (1 - \cos p_a)$$



$$\mathcal{G}(\mathbf{p})|_{\mu_5=0} = \Big(\sum_k \gamma^k g_k(\mathbf{p}) - i m(\mathbf{p})\Big)^{-1}$$

$$g_5[p] = m[p] \qquad \hat{g} = \frac{g}{\sqrt{g^a g^a}}$$

$$\tilde{\mathcal{N}}_5 = \frac{3}{4\pi^2 4!} \epsilon_{abcde} \int \hat{g}^a \, d\hat{g}^b \wedge d\hat{g}^c \wedge d\hat{g}^d \wedge d\hat{g}^e$$

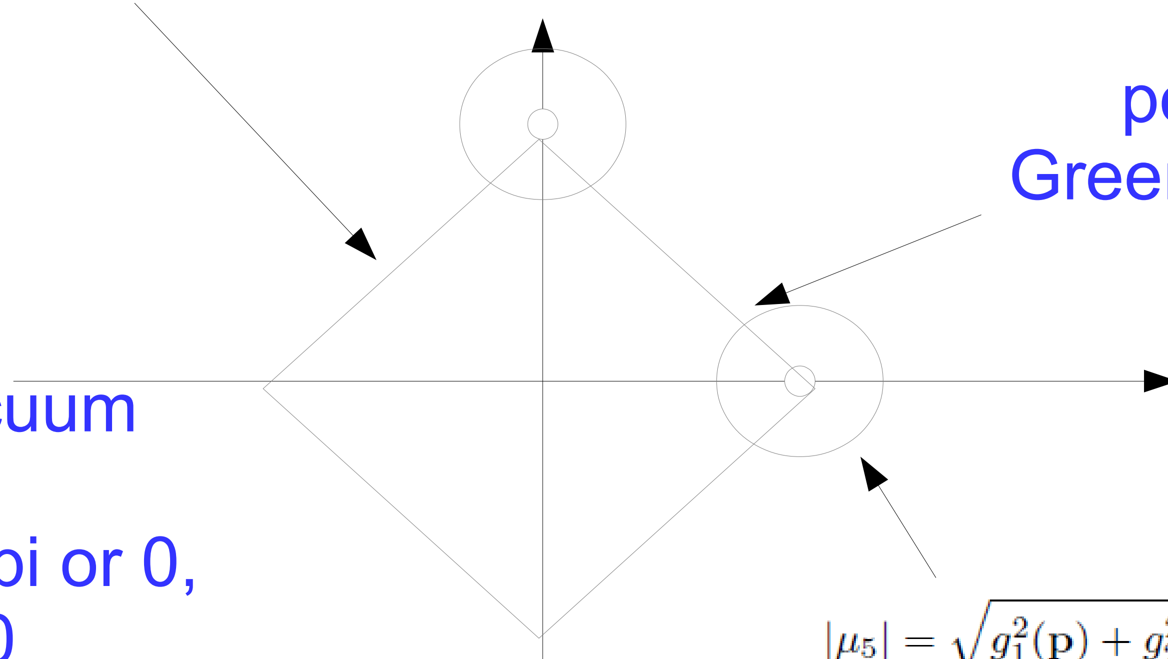
3+1 D Chiral Magnetic Effect

nontrivial example Wilson fermions with $m^{(0)} = -2.$

$$\mathcal{G}(\mathbf{p}) = \Big(\sum_k \gamma^k g_k(\mathbf{p}) + i\gamma^4\gamma^5\mu_5 - im(\mathbf{p}) \Big)^{-1}$$

$$g_k(\mathbf{p}) = \sin p_k, \quad m(\mathbf{p}) = m^{(0)} + \sum_{a=1,2,3,4} (1 - \cos p_a)$$

the zeros of m(p) form the curves (p_3=p_4=0)



pole of the Green function

Fermi lines

Marginal vacuum

If p_4 is not pi or 0,

N_3(p_4) = 0

$$|\mu_5| = \sqrt{g_1^2(\mathbf{p}) + g_2^2(\mathbf{p}) + g_3^2(\mathbf{p})}$$

3+1 D Chiral Magnetic Effect 3D Dirac semimetal

nontrivial example with
Wilson fermions

$$\mathcal{M}_4 = -\frac{i}{2}\int dp^4 \tilde{\mathcal{N}}_3(p^4),$$

$$\tilde{\mathcal{N}}_3(p^4) = \frac{1}{24\pi^2}\epsilon_{ijk4}\mathrm{Tr}\int_\Omega d^3p\Big(\mathcal{G}\partial^i\mathcal{G}^{-1}\Big)\Big(\mathcal{G}\partial^j\mathcal{G}^{-1}\Big)\Big(\mathcal{G}\partial^k\mathcal{G}^{-1}\Big)$$

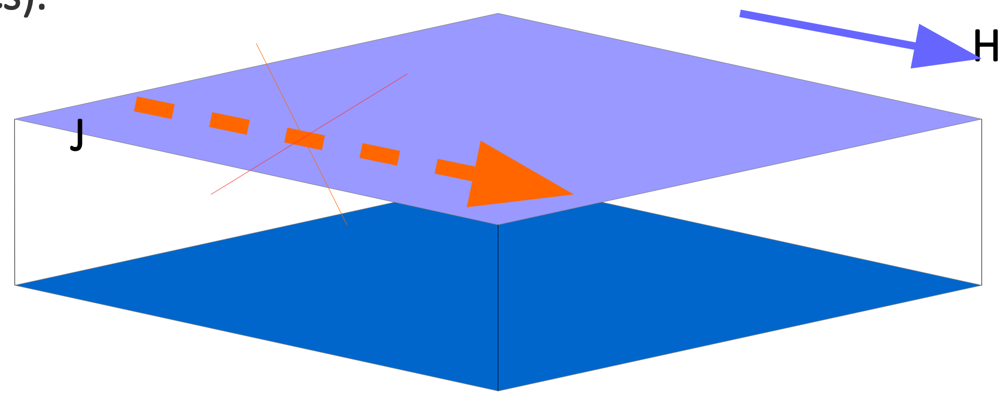
If we regularize the
Integral as

$$\int = \lim_{\epsilon\to 0}\Big(\int_{-\infty}^{-\epsilon} + \int_{\epsilon}^{\infty}\Big)$$

Then M_4 → 0

At finite temperature we use the sum over the Matsubara frequencies $\omega_n = T\pi(2n+1) \neq 0$
And the answer is the same

We considered lattice models with both massive qnd massless fermions that describe lattice regularized quantum field theory or the insulators and Dirac semimetals whose excitations are described by massive/massless Dirac action (in solid state physics).



J = M_4 /2pi^2 H   M_4 = 0 as long as mu_5 is nonzero

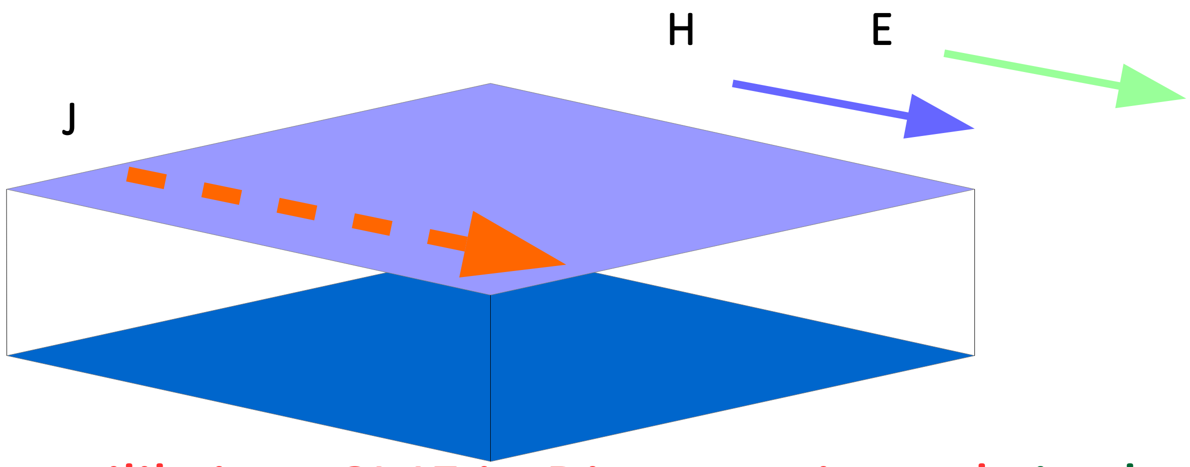
Chiral imbalance is described by the appearance of the chiral chemical potential

Green function (without external magnetic field) is:

$$\mathcal{G}(\mathbf{p}) = \Big(\sum_k \gamma^k g_k(\mathbf{p}) + i\gamma^4\gamma^5\mu_5 - im(\mathbf{p})\Big)^{-1}$$

There is no equilibrium static bulk CME

IN WHICH FORM THE CME MAY SURVIVE ?



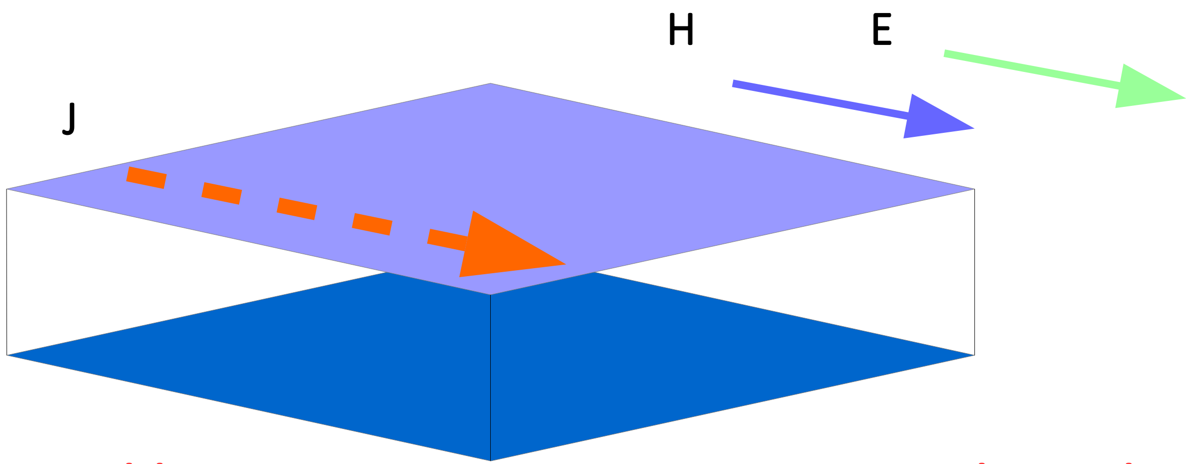
1) nonequilibrium CME in Dirac semimetals in the presence of external magnetic and electric field the chiral anomaly produces chiral imbalance

this production requires energy taken from the job performed by the electic field.

This assumes existence of electric current j

JE = the energy created while pumping pairs from vacuum => J = ?

IN WHICH FORM THE CME MAY SURVIVE ?



2) nonequilibrium CME in Dirac semimetals in the presence of emergent magnetic field (say, due to the dislocations)

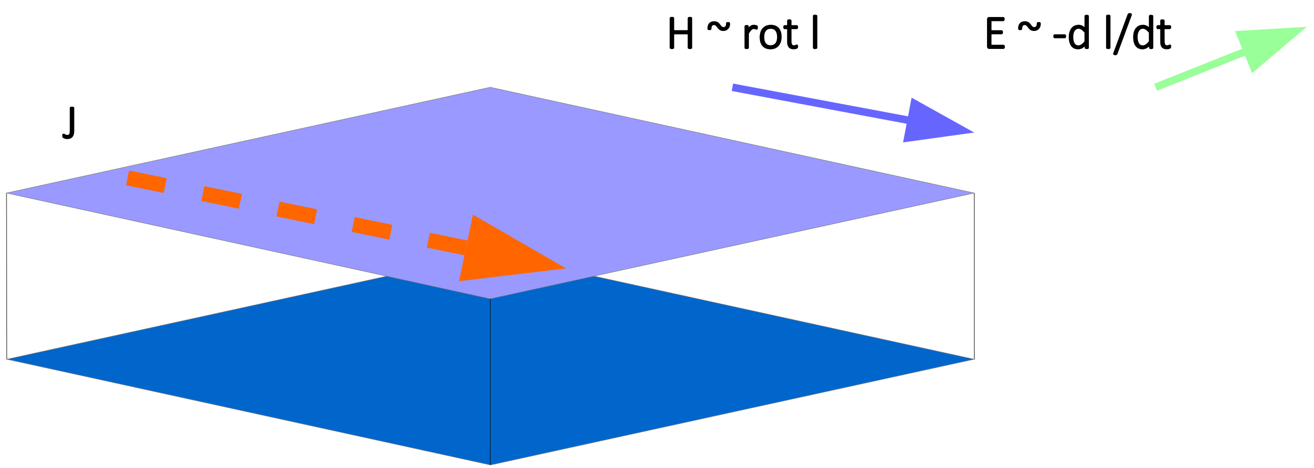
the chiral anomaly produces chiral Imbalance

this production requires energy taken from the job performed by the electic field.

This assumes existence of electric current j

JE = energy created while pumping the pairs from vacuum =>
J = ?

IN WHICH FORM THE CME MAY SURVIVE ?

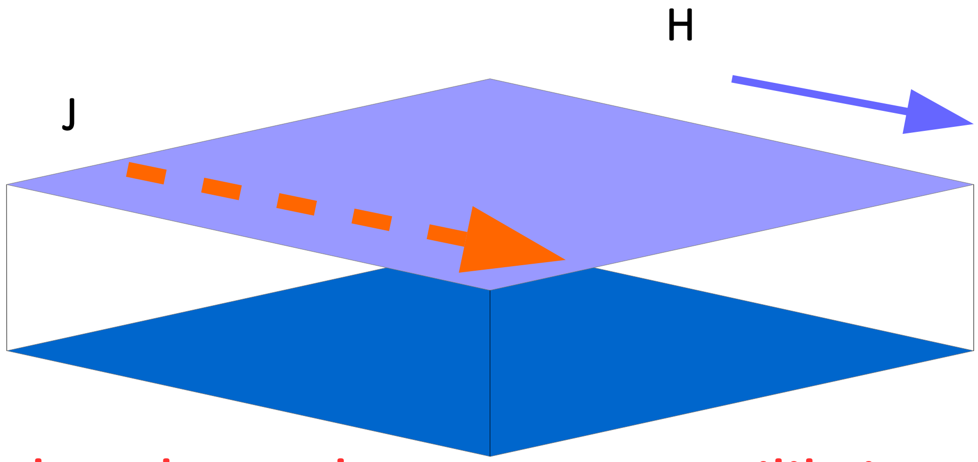


3) CME in He3-A, where mu_5 ~ l (v_n-v_s)

The applied technique for the calculation of the CME current does not work here because :

- the problem is not equilibrium
- the gauge field is emergent rather than real

IN WHICH FORM THE CME MAY SURVIVE ?



4) Quark – gluon plasma : nonequilibrium CME contributions to the kinetic equations in the presence of the chiral imbalance?

Chiral imbalance that is described by chiral density rather than the chiral chemical potential ?

3+1 D Anomalous Quantum Hall effect 3D top. insulator

We obtain for the first time

$$j^k_{Hall} = \frac{1}{4\pi^2}\mathcal{M}'_l\,\epsilon^{jkl}E_j$$

$$\mathcal{M}'_l \;=\; \frac{1}{3!\,4\pi^2}\,\epsilon_{ijkl}\int d^4p\,\mathrm{Tr}\Big[\mathcal{G}\frac{\partial\mathcal{G}^{-1}}{\partial p_i}\frac{\partial\mathcal{G}}{\partial p_j}\frac{\partial\mathcal{G}^{-1}}{\partial p_k}\Big]$$

In the particular case of the non — interacting system it is reduced to

$$\mathcal{G}^{-1} = i\omega - \hat{H} \qquad\qquad \mathcal{M}'_l \;=\; \frac{\epsilon^{ijl}}{4\pi}\sum_{\mathrm{occupied}}\int d^3p\,\mathcal{F}_{ij}$$

Berry curvature $\mathcal{F}_{ij} \;=\; \partial_i\mathcal{A}_j - \partial_j\mathcal{A}_i \qquad \mathcal{A}_j = i\langle k,\vec{p}|\partial_j|k,\vec{p}\rangle$

3+1 D Anomalous Quantum Hall effect 3D top. insulator

We obtain for the first time

$$j^k_{Hall} = \frac{1}{4\pi^2}\,\mathcal{M}'_l\,\epsilon^{jkl}E_j$$

$$\mathcal{M}'_l = \frac{1}{3!\,4\pi^2}\,\epsilon_{ijkl}\int d^4p\,\mathrm{Tr}\Big[\mathcal{G}\frac{\partial\mathcal{G}^{-1}}{\partial p_i}\frac{\partial\mathcal{G}}{\partial p_j}\frac{\partial\mathcal{G}^{-1}}{\partial p_k}\Big]$$

2x2 Green function

$$\mathcal{G}^{-1}(\mathbf{p}) = i\sigma^3\Big(\sum_k \sigma^k g_k(\mathbf{p}) - i g_4(\mathbf{p})\Big)$$

Sum over points, where g_k=0 (k=1,2,3)

$$\hat{g}_k = \frac{g_k}{g}$$

$$\mathcal{M}'_j = -\frac{1}{2}\sum_l \int_{y_l(s)} \mathrm{sign}(g_4(y_l))\,\mathrm{Res}\,(y_l)dp_j$$

$$g = \sqrt{\sum_{k=1,2,3,4} g_k^2}$$

$$\mathrm{Res}\,(y_l) = \frac{1}{8\pi}\epsilon^{ijk}\int_{\Sigma(y_l)} \hat{g}_i d\hat{g}_j \wedge d\hat{g}_k$$

3+1 D Anomalous Quantum Hall effect 3D top. insulator

Example $\mathcal{G}^{-1} = i\omega - \hat{H}$ $j^k_{Hall} = \frac{1}{4\pi^2} \mathcal{M}'_l \, \epsilon^{jkl} E_j$

$$H = \sin p_1 \, \sigma^2 - \sin p_2 \, \sigma^1 - (m^{(0)} - \gamma \cos p_3 + \sum_{i=1,2} (1 - \cos p_i)) \, \sigma^3$$

$\gamma < 1$, and $m^{(0)} \in (-2+\gamma, -\gamma)$

$$\hat{g}_4(\mathbf{p}) = \frac{(m^{(0)} - \gamma \cos p_3 + \sum_{i=1,2}(1 - \cos p_i))}{\sqrt{(m^{(0)} - \gamma \cos p_3 + \sum_{i=1,2}(1 - \cos p_i))^2 + \sin^2 p_1 + \sin^2 p_2 + \omega^2}}$$

$$\begin{aligned}
\hat{g}_4(\mathbf{p}) &= 0, \quad \mathbf{p} \in \partial\mathcal{M} \quad (\omega \to \pm\infty) \\
\hat{g}_4(\mathbf{p}) &= -1, \quad \hat{g}_i(\mathbf{p}) = 0 \quad (k = 1,2,3), \quad \mathbf{p} = (0,0,p_3,0), \quad p_3 \in (-\pi,\pi) \\
\hat{g}_4(\mathbf{p}) &= 1, \quad \hat{g}_i(\mathbf{p}) = 0 \quad (k = 1,2,3), \quad \mathbf{p} = (0,\pi,p_3,0), \quad p_3 \in (-\pi,\pi) \\
\hat{g}_4(\mathbf{p}) &= 1, \quad \hat{g}_i(\mathbf{p}) = 0 \quad (k = 1,2,3), \quad \mathbf{p} = (\pi,0,p_3,0), \quad p_3 \in (-\pi,\pi) \\
\hat{g}_4(\mathbf{p}) &= 1, \quad \hat{g}_i(\mathbf{p}) = 0 \quad (k = 1,2,3), \quad \mathbf{p} = (\pi,\pi,p_3,0), \quad p_3 \in (-\pi,\pi)
\end{aligned}$$

$$\mathcal{M}'_3 = \frac{2\pi}{2} - \frac{2\pi}{2}(-1) - \frac{2\pi}{2}(-1) - \frac{2\pi}{2} = 2\pi \qquad j^k_{Hall} = \frac{1}{2\pi\, a} \epsilon^{jk3} E_j$$

3+1 D Anomalous Quantum Hall effect 3D top. insulator

We obtain for the first time

$$j^k_{Hall} = \frac{1}{4\pi^2}\mathcal{M}'_l\,\epsilon^{jkl}E_j$$

$$\mathcal{M}'_l \;=\; \frac{1}{3!\,4\pi^2}\,\epsilon_{ijkl}\int d^4p\,\mathrm{Tr}\Big[\mathcal{G}\frac{\partial\mathcal{G}^{-1}}{\partial p_i}\frac{\partial\mathcal{G}}{\partial p_j}\frac{\partial\mathcal{G}^{-1}}{\partial p_k}\Big]$$

4x4 Green function

$$\mathcal{G}(\mathbf{p}) = \Big(\sum_k \gamma^k g_k(\mathbf{p}) + \gamma^5 g_5(\mathbf{p}) + \gamma^3\gamma^5 b(\mathbf{p})\Big)^{-1}$$

The sum over
two 2x2 systems

$$\mathcal{M}'_l \;=\; \mathcal{M}'_{l,+} + \mathcal{M}'_{l,-}$$

$$\mathcal{G}^{-1}(\mathbf{p}) = \sigma^3\Big(\sum_k \sigma^k g'_k(\mathbf{p}) - i g'_4(\mathbf{p})\Big)$$

$$g'_1 = g_2(\mathbf{p}),\quad g'_2 = -g_1(\mathbf{p}),\quad g'_3 = g_4(\mathbf{p}),\quad g'_4 = \mp\Big(\sqrt{g_3^2(\mathbf{p}) + g_5^2(\mathbf{p})} \pm b(\mathbf{p})\Big)$$

3+1 D Anomalous Quantum Hall effect 3D top. insulator

We obtain for the first time

$$j^k_{Hall} = \frac{1}{4\pi^2} \mathcal{M}'_l \, \epsilon^{jkl} E_j$$

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4x4 Green function

$$\mathcal{G}(\mathbf{p}) = \Big(\sum_k \gamma^k g_k(\mathbf{p}) - i g_5(\mathbf{p}) + \gamma^3 \gamma^5 b(\mathbf{p})\Big)^{-1}$$

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3+1 D Anomalous Quantum Hall effect 3D top. insulator

Example $\mathcal{G}^{-1} = i\omega - \hat{H}$ $j^k_{Hall} = \frac{1}{4\pi^2}\mathcal{M}'_l\,\epsilon^{jkl}E_j$

$$\mathcal{G}(\mathbf{p}) = \Big(\sum_k \gamma^k g_k(\mathbf{p}) + \gamma^5 g_5(\mathbf{p}) + \gamma^3\gamma^5 b(\mathbf{p})\Big)^{-1} \qquad g_3^{(0)} > 1,\ m^{(0)} > g_3^{(0)} - 1$$

$$g_1(\mathbf{p}) = -\sin p_2, \quad g_2(\mathbf{p}) = \sin p_1, \quad g_3(\mathbf{p}) = g_3^{(0)} + \sin p_3$$
$$g_4(\mathbf{p}) = \omega, \quad g_5(\mathbf{p}) = m^{(0)} + \sum_{a=1,2}(1-\cos p_a), \quad b = const$$

$$\sqrt{(g_3^{(0)}+1)^2 + (m^{(0)})^2} < b$$

$$\sqrt{(g_3^{(0)}-1)^2 + (m^{(0)}+2)^2} > b$$

$$\mathcal{M}'_3 = \frac{2\pi}{2} - \frac{2\pi}{2}(-1) - \frac{2\pi}{2}(-1) - \frac{2\pi}{2} = 2\pi \qquad j^k_{Hall} = \frac{1}{2\pi\,a}\epsilon^{jk3}E_j$$

3+1 D Anomalous Quantum Hall effect for Weyl semimetal

Example $\mathcal{G}^{-1} = i\omega - \hat{H}$ $j^k_{Hall} = \frac{1}{4\pi^2} \mathcal{M}'_l \, \epsilon^{jkl} E_j$

$$\mathcal{G}(\mathbf{p}) = \Big(\sum_k \gamma^k g_k(\mathbf{p}) + \gamma^5 g_5(\mathbf{p}) + \gamma^3 \gamma^5 b(\mathbf{p}) \Big)^{-1}$$

$$g_1(\mathbf{p}) = -\sin p_2, \quad g_2(\mathbf{p}) = \sin p_1, \quad g_3(\mathbf{p}) = g_3^{(0)} + \sin p_3$$
$$g_4(\mathbf{p}) = \omega, \quad g_5(\mathbf{p}) = m^{(0)} + \sum_{a=1,2} (1 - \cos p_a), \quad b = const$$

$$g_3^{(0)} > \sqrt{b^2 - (m^{(0)})^2} > g_3^{(0)} - 1 > 0 \qquad \sqrt{(g_3^{(0)} + \sin \beta_\pm)^2 + (m^{(0)})^2} = b$$

Two Fermi points $\mathbf{K}_\pm = (0, 0, \beta_\pm, 0)$

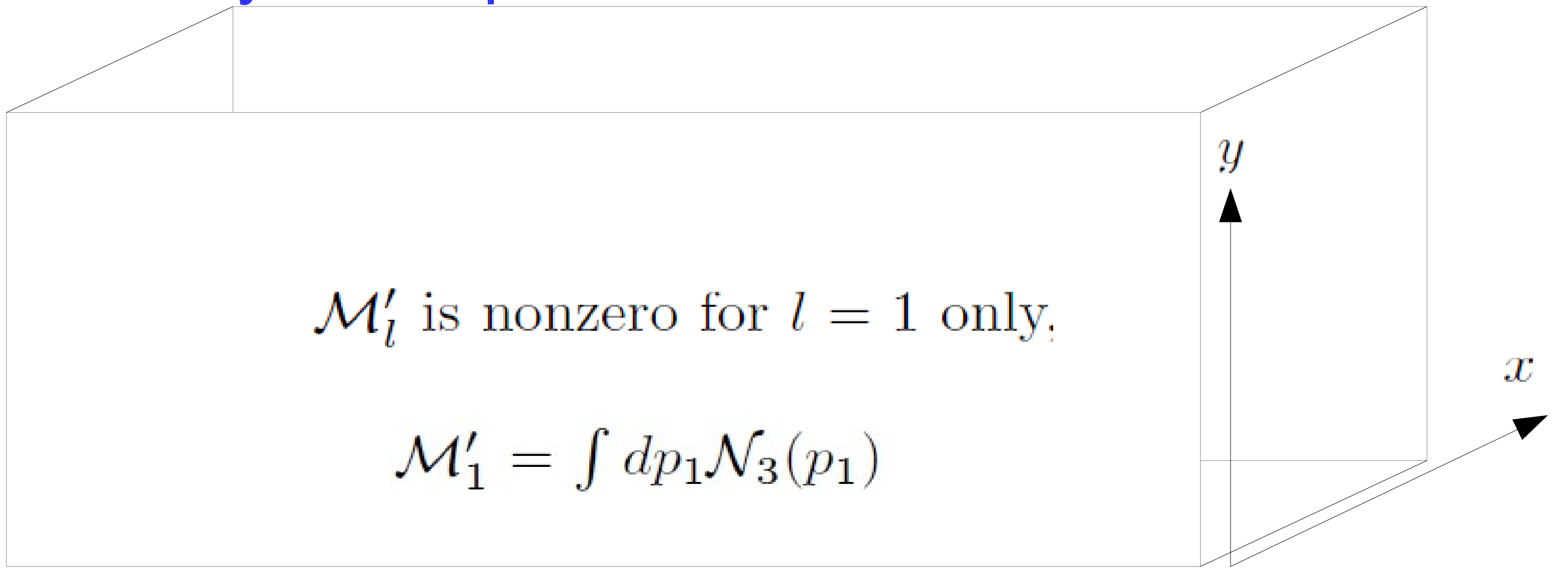
$$j^k_{Hall} = \frac{\beta_+ - \beta_-}{4\pi^2} \, \epsilon^{jk3} E_j$$

the same expression may be obtained using effective continous QFT

3+1 D AQH, top. insulator

$$j^k_{Hall} = \frac{1}{4\pi^2}\mathcal{M}'_l\,\epsilon^{jkl}E_j$$

Bulk — boundary correspondence



$$\mathcal{N}_3(p_1) \;=\; \frac{1}{3!\,4\pi^2}\,\epsilon_{ijkl}\int dp_2 dp_3 dp_4\,\mathrm{Tr}\left[\tilde{G}^{(0)}\frac{\partial(\tilde{G}^{(0)})^{-1}}{\partial p_i}\frac{\partial \tilde{G}^{(0)}}{\partial p_j}\frac{\partial(\tilde{G}^{(0)})^{-1}}{\partial p_k}\right]$$

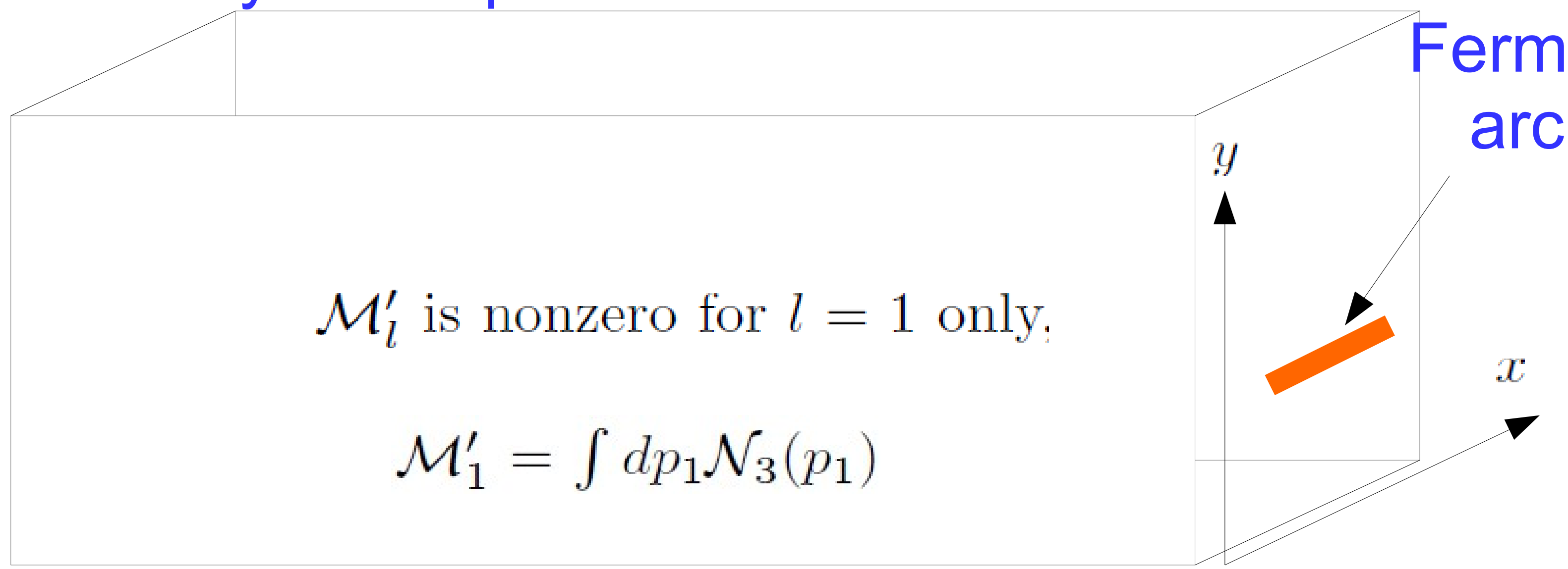
Index theorem: at each value of p_1 the jump of N_3 is equal to the number of gapless chiral boundary modes.

We have N_3 Fermi lines on the xy and xz boundaries and N_3 Fermi points on the yz boundary

3+1 D AQH, Weyl semimetal

Bulk — boundary correspondence

$$j^k_{Hall} = \frac{1}{4\pi^2}\mathcal{M}'_l\,\epsilon^{jkl}E_j$$



$\mathcal{M}'_l$ is nonzero for $l = 1$ only.

$$\mathcal{M}'_1 = \int dp_1 \mathcal{N}_3(p_1)$$

$$\mathcal{N}_3(p_1) = \frac{1}{3!\,4\pi^2}\epsilon_{ijkl}\int dp_2 dp_3 dp_4\,\mathrm{Tr}\Big[\tilde{G}^{(0)}\frac{\partial(\tilde{G}^{(0)})^{-1}}{\partial p_i}\frac{\partial \tilde{G}^{(0)}}{\partial p_j}\frac{\partial(\tilde{G}^{(0)})^{-1}}{\partial p_k}\Big]$$

Index theorem: at each value of p_1 the jump of N_3 is equal to the number of gapless chiral boundary modes.

We have N_3 Fermi arcs on the xy and xz boundaries that

Conclusions

1. The formalism of Wigner transformation has been applied to the Green functions defined in compact momentum space.

2. Using derivative expansions applied to the Wigner transform of the Green function we derive

$$j^k_{Hall} = \frac{1}{2\pi}\tilde{\mathcal{N}}_3\, \epsilon^{ki} E_i \quad \text{and} \quad j^k_{Hall} = \frac{1}{4\pi^2}\mathcal{M}^\prime_l\, \epsilon^{jkl} E_j$$

3. The technique for the calculation of these top. invariants is developed and applied to AQHE in topological insulators and Weyl semimetals.

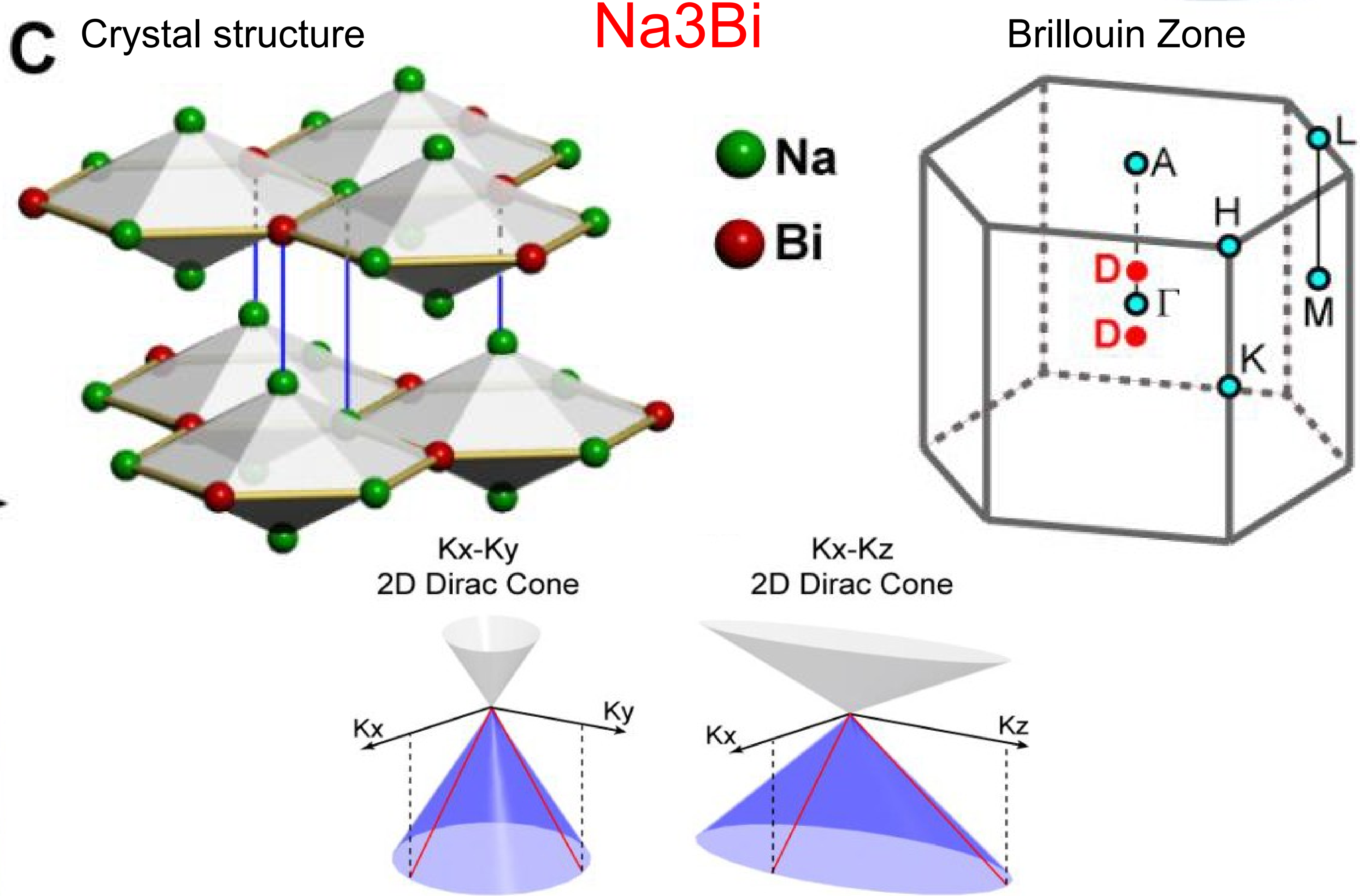
4. Bulk – boundary correspondence in terms of the Wigner transform of the Green function allows to explain the existence of the Fermi lines on the boundaries of the topological insulators with AQHE and Fermi arcs on the boundaries of the Weyl semimetals.

5. We derive for the equilibrium static bulk chiral magnetic effect

$$j^{(1)k}(\mathbf{R}) = \frac{1}{4\pi^2}\epsilon^{ijkl}\mathcal{M}_l A_{ij}(\mathbf{R})$$

This top. invariant vanishes for the Dirac semimetals and for the lattice regularized QFT both with and without nonzero mass of the fermions.

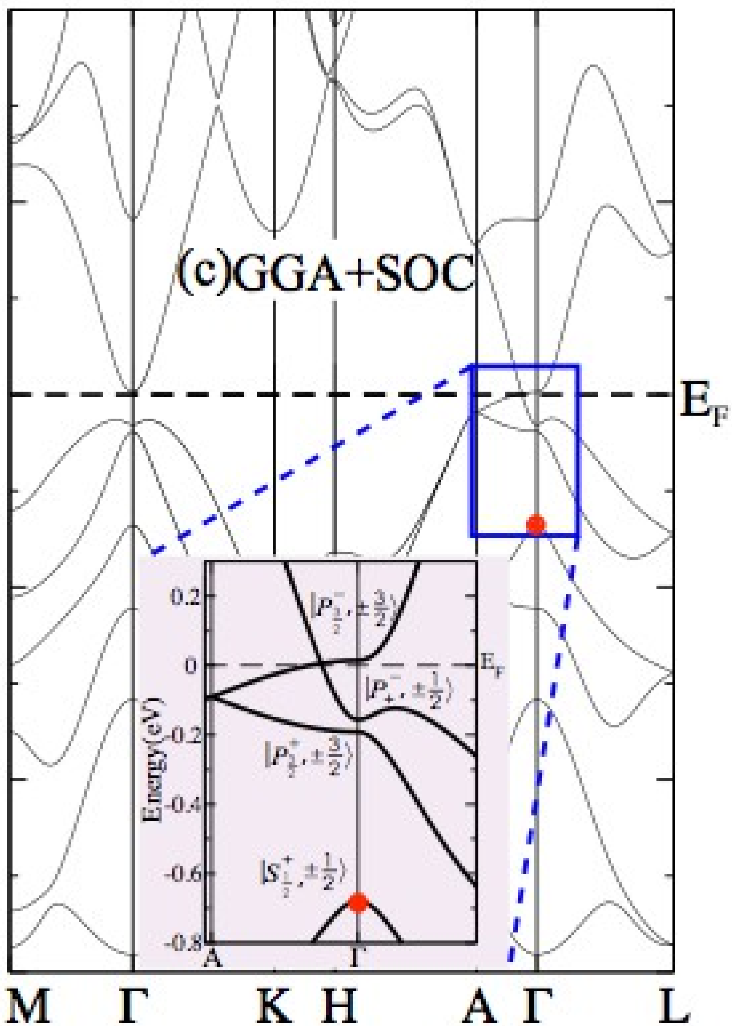
Na3Bi



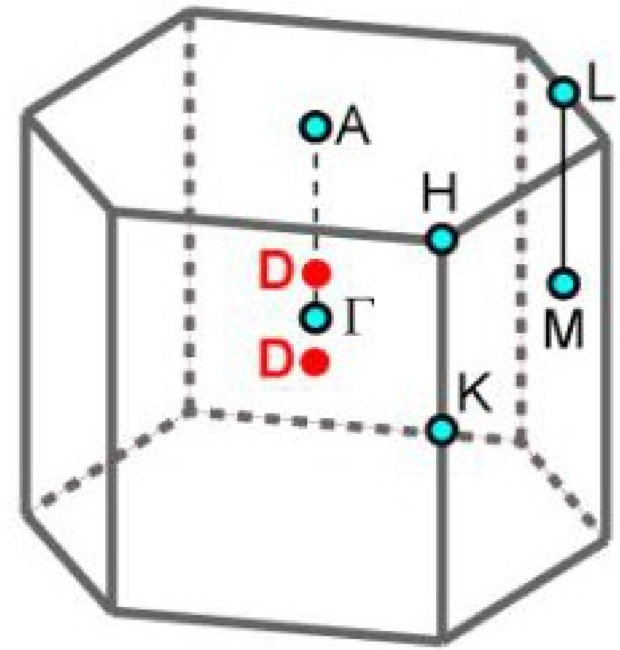
K. Liu et al., Science (2014) Vol. 343 no. 6173 pp. 864-867 DOI: 10.1126/science.1245085, arXiv:1310.039

Na3Bi

Band structure



Brillouin Zone

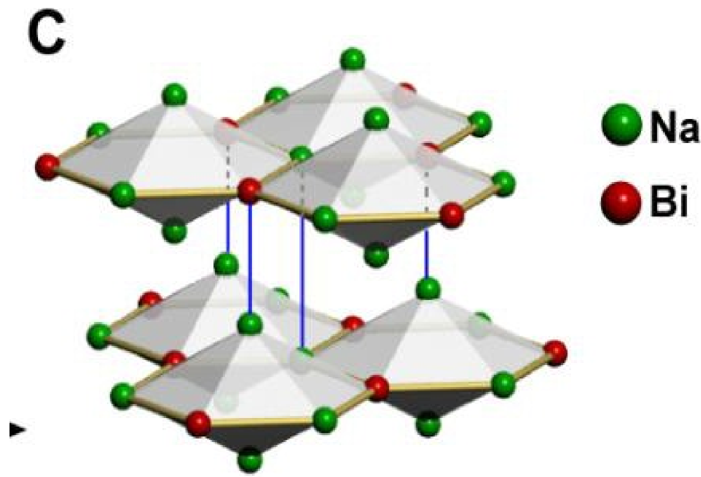


Na3Bi

Effective Hamiltonian

$$|S^{\pm}\rangle = \frac{1}{\sqrt{2}}(|Na,s\rangle \pm |Na',s\rangle),$$

$$|P^{\pm}_{\alpha}\rangle = \frac{1}{\sqrt{2}}(|Bi,p_\alpha\rangle \mp |Bi',p_\alpha\rangle),$$



$$|S^{+}_{\frac{1}{2}},\tfrac{1}{2}\rangle,\ |P^{-}_{\frac{3}{2}},\tfrac{3}{2}\rangle,\ |S^{+}_{\frac{1}{2}},-\tfrac{1}{2}\rangle,\ |P^{-}_{\frac{3}{2}},-\tfrac{3}{2}\rangle)$$

$$H_\Gamma(\mathbf{k}) = \epsilon_0(\mathbf{k}) + \begin{pmatrix} M(\mathbf{k}) & Ak_+ & 0 & B^*(\mathbf{k}) \\ Ak_- & -M(\mathbf{k}) & B^*(\mathbf{k}) & 0 \\ 0 & B(\mathbf{k}) & M(\mathbf{k}) & -Ak_- \\ B(\mathbf{k}) & 0 & -Ak_+ & -M(\mathbf{k}) \end{pmatrix}$$

$$\epsilon_0(\mathbf{k}) = C_0 + C_1 k_z^2 + C_2(k_x^2 + k_y^2),\ k_\pm = k_x \pm i k_y$$

$$M(\mathbf{k}) = M_0 - M_1 k_z^2 - M_2(k_x^2 + k_y^2)$$

Na3Bi

Effective Hamiltonian $\quad |S^+_{\frac{1}{2}}, \frac{1}{2}\rangle, |P^-_{\frac{3}{2}}, \frac{3}{2}\rangle, |S^+_{\frac{1}{2}}, -\frac{1}{2}\rangle, |P^-_{\frac{3}{2}}, -\frac{3}{2}\rangle)$

$$H_\Gamma(\mathbf{k}) = \epsilon_0(\mathbf{k}) + \begin{pmatrix} M(\mathbf{k}) & Ak_+ & 0 & B^*(\mathbf{k}) \\ Ak_- & -M(\mathbf{k}) & B^*(\mathbf{k}) & 0 \\ 0 & B(\mathbf{k}) & M(\mathbf{k}) & -Ak_- \\ B(\mathbf{k}) & 0 & -Ak_+ & -M(\mathbf{k}) \end{pmatrix}$$

$$\epsilon_0(\mathbf{k}) = C_0 + C_1 k_z^2 + C_2(k_x^2 + k_y^2),\ k_\pm = k_x \pm i k_y$$

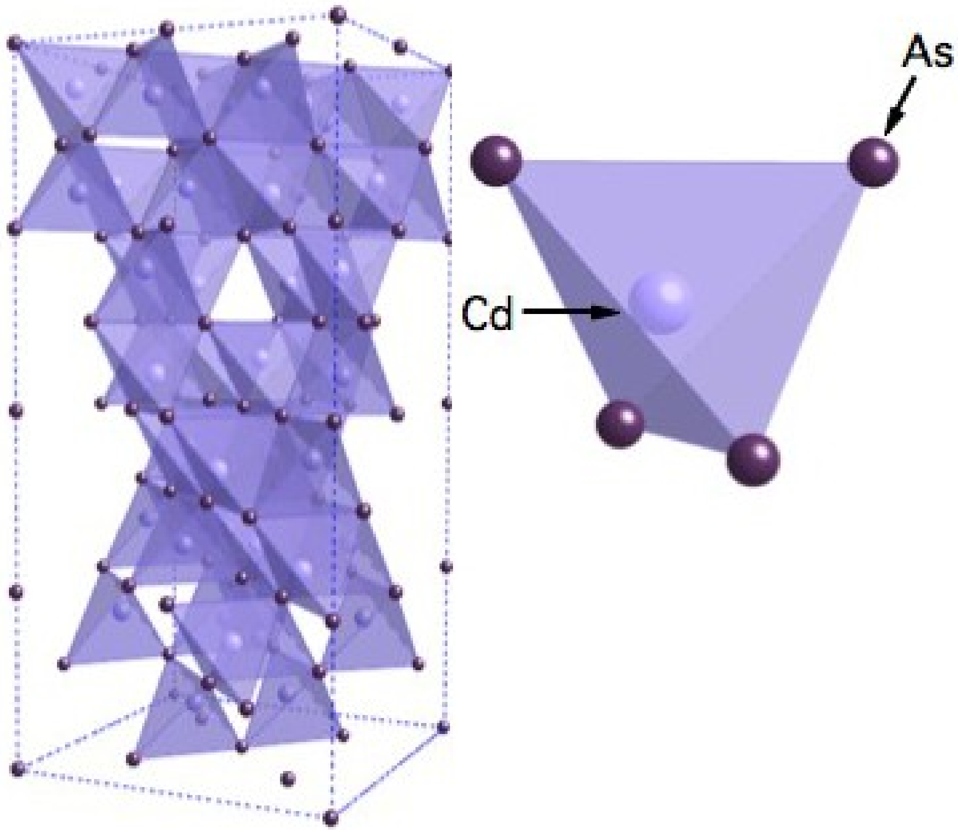
$$M(\mathbf{k}) = M_0 - M_1 k_z^2 - M_2(k_x^2 + k_y^2)$$

$$E(\mathbf{k}) = \epsilon_0(\mathbf{k}) \pm \sqrt{M(\mathbf{k})^2 + A^2 k_+ k_- + |B(\mathbf{k})|^2}$$

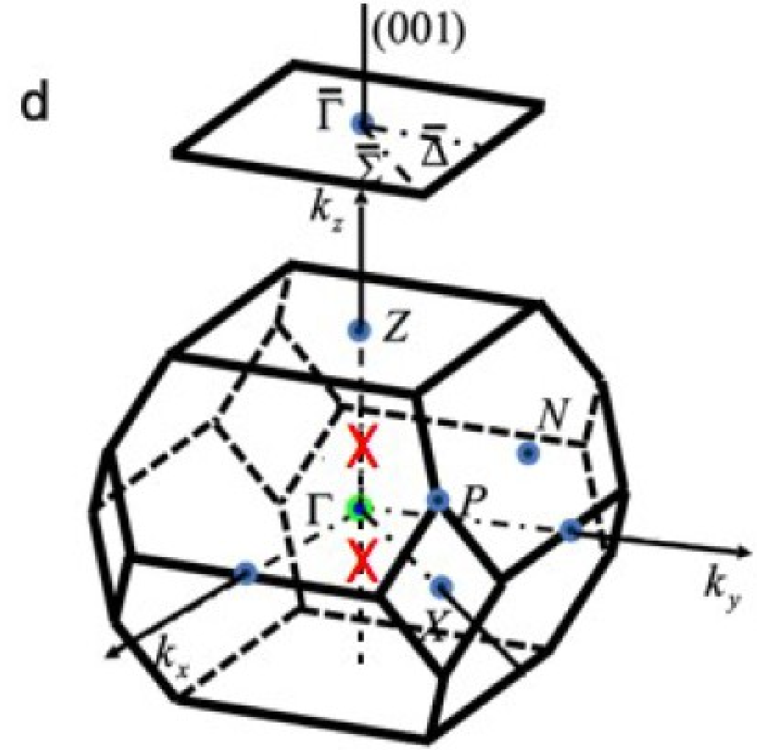
Dirac points $\quad \mathbf{k}^c = (0,\ 0,\ k_z^c = \pm\sqrt{\frac{M_0}{M_1}})$

Cd3As2

Crystal structure



Brillouin Zone

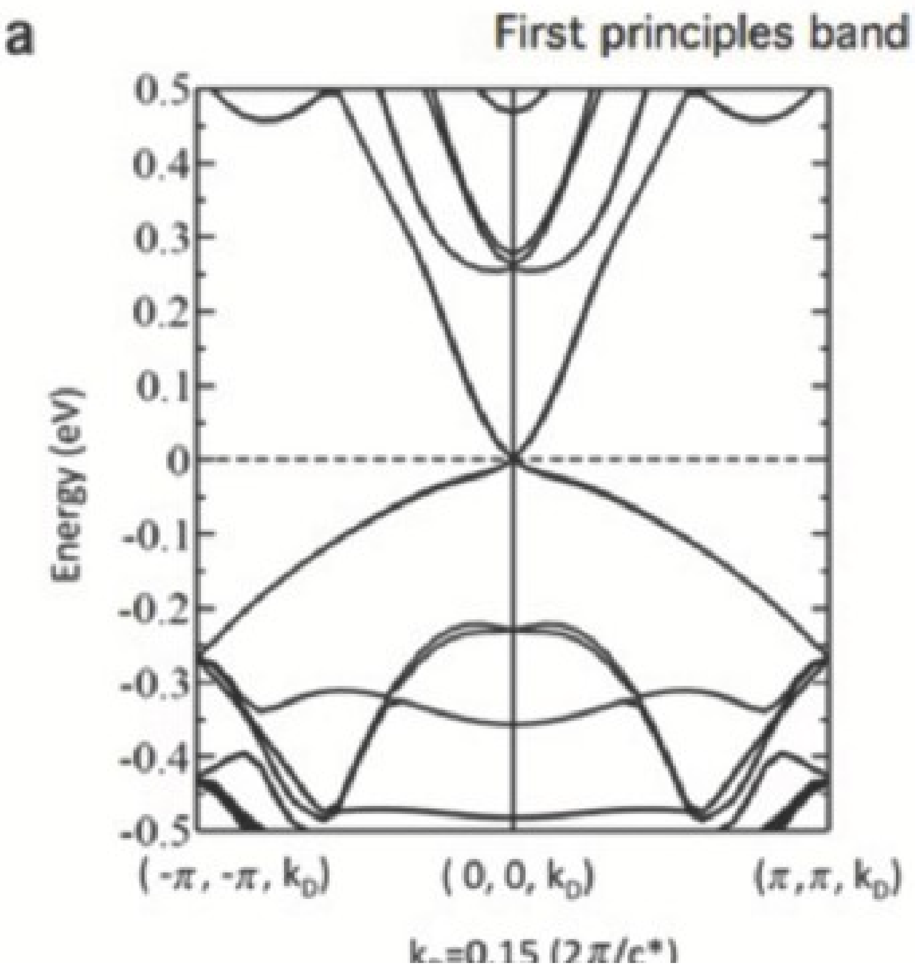


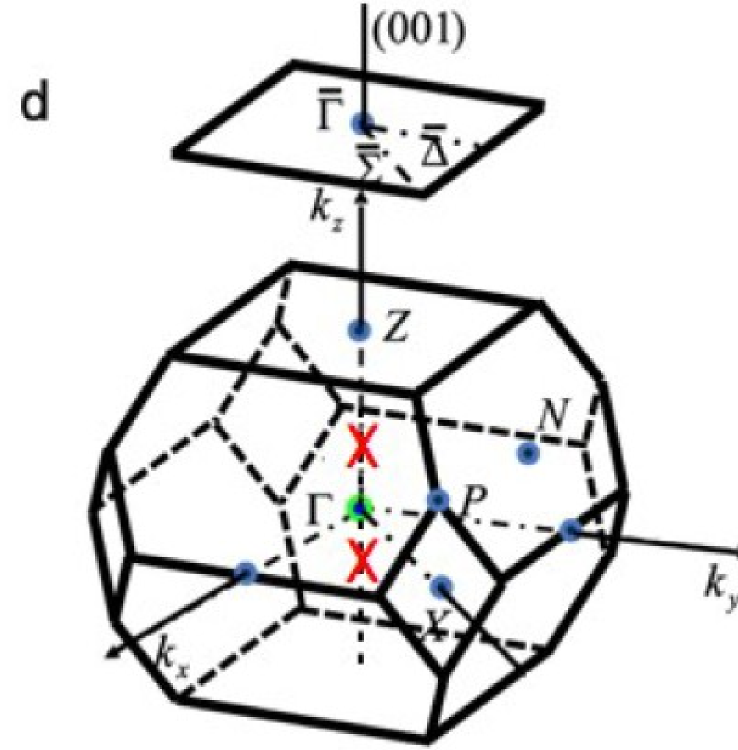
M. Neupane et al.,Nature Commun. 05, 3786 (2014), DOI: 10.1038/ncomms4786, arXiv:1309.7892

S. Borisenko et al., Phys. Rev. Lett. 113, 027603 (2014), DOI: 10.1103/PhysRevLett.113.027603,arXiv:1309.7978

Cd3As2

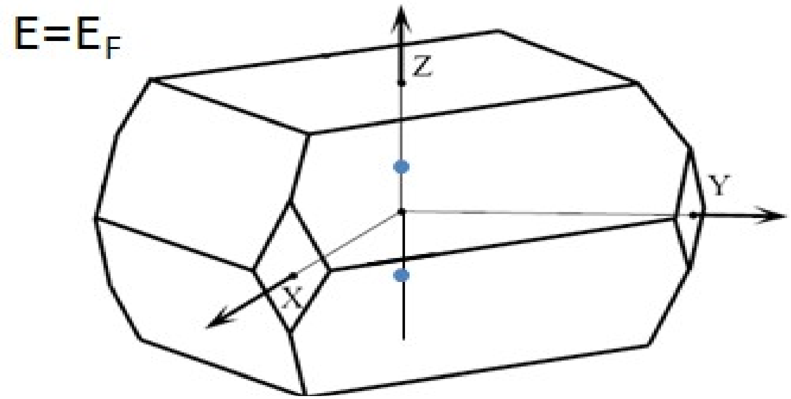
Brillouin Zone

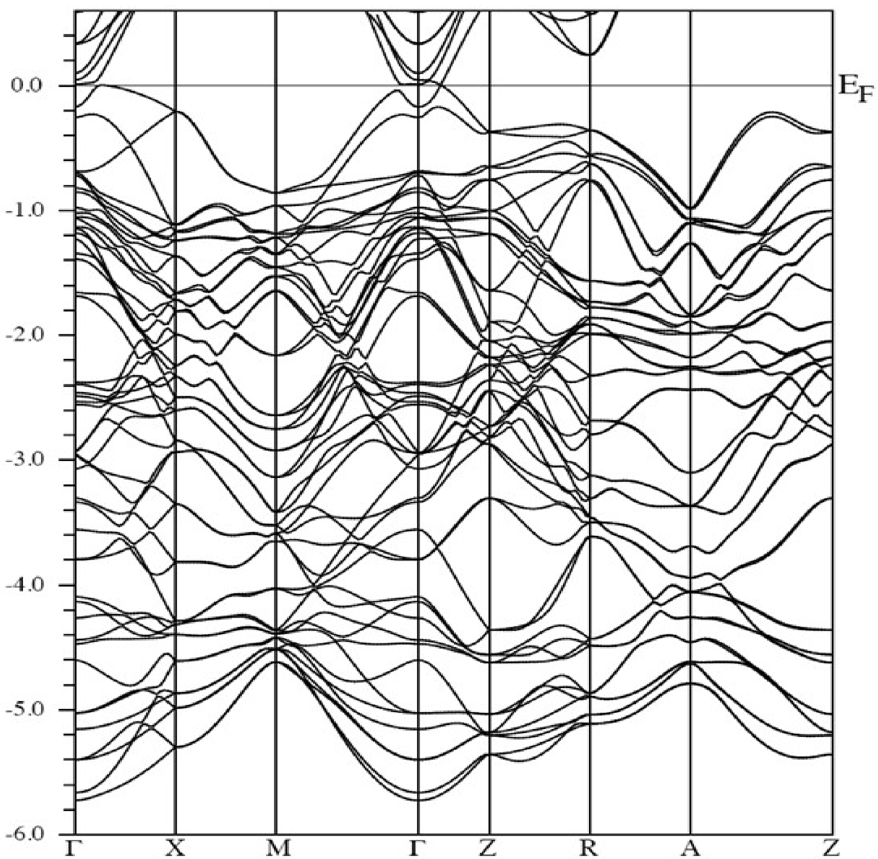


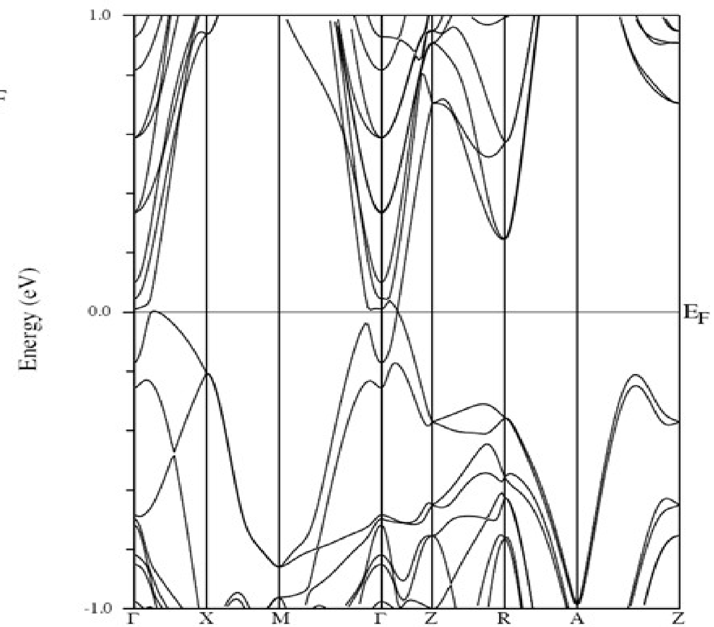


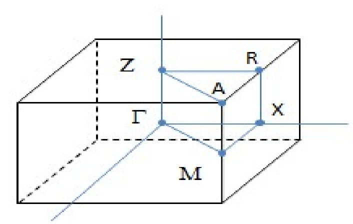
Cd3As2

Brillouin Zone









Cd3As2

Brillouin Zone