Classical and Quantum Black Holes
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Black hole thermodynamics from a variational principle:

Asymptotically conical background

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Ok Song An, M.C., Ioannis Papadimitriou, 1602.0150, JHEP03 (2016)086

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Asymptotically flat or anti-deSitter (cosmological constant $\Lambda < 0$) spacetimes have been studied extensively, but new asymptotic backgrounds, asymptotically supported by matter fields, attracted attention only relatively recently.

Examples:

ii. Flux vacua of string compactification, dual to certain supersymmetric quantum field theories (Klebanov-Strassler)

ii. Lifshitz black holes, dual to certain non-relativistic quantum mechanical condensed systems

iii. `Subtracted geometry', proposed to addressed microscopics of certain non-extremal black holes.

Understanding the mesoscopic (geometric) properties of black holes with such exotic asymptotics

→ a first step towards their microscopic description.

While black hole entropy & temperature is not sensitive to the asymptotics, as they are intrinsically connected with the horizon, conserved charges and the free energy depend on the spacetime asymptotics and receive contribution from asymptotic matter fields.

E.g., large distance divergences, plaguing conserved charges and free energy may not be always remedied by, say, Komar integral or background subtraction.

Main motivation: formulating well posed variational problem for new asymptotic geometries (following lessons from AdS geometries Heningson,Skenderis'98; Balasubramanian,Kraus'99; deBoer,Verlinde^2'99,…): to determine conserved charges & black hole thermodynamics.

Approach general (does not rely on the specific theory or its asymptotic solutions), but apply to concrete example of subtracted geometry.

Summay: variational problem for new asymptotic geometries

Achieved through an algorithmic procedure:

- integration constants, parameterizing solutions of the eqs. of motion, separated into `normalizable' - free to vary & 'non-normalizable' modes – fixed

- non-normalizable modes – fixed only up to transformations induced by local symmetries of the bulk theory (radial diffeomorphisms & gauge transformations)

- covariant boundary term, S_{ct}, to the bulk action - determined by solving asymptotically the radial Hamilton-Jacobi eqn. → total action $S+S_{ct}$ independent of the radial coordinate

Skenderis,Papadimitriou'04,Papadimitriou'05

- first class constraints of Hamiltonian formal. lead to conserved charges associated with Killing vectors. [S_{ct} 'renormalizes' the phase space: charges are independent of the radial coordinate.]

- Conserved charge automatically satisfy the first law of thermodynamics

Outline:

I. Microscopics of Black Hole Entropy & String Theory

II. Motivation and review of subtracted geometry:

Classical geometry insights

Summary of quantum aspects

Key Issue in Black Hole Physics:

How to relate

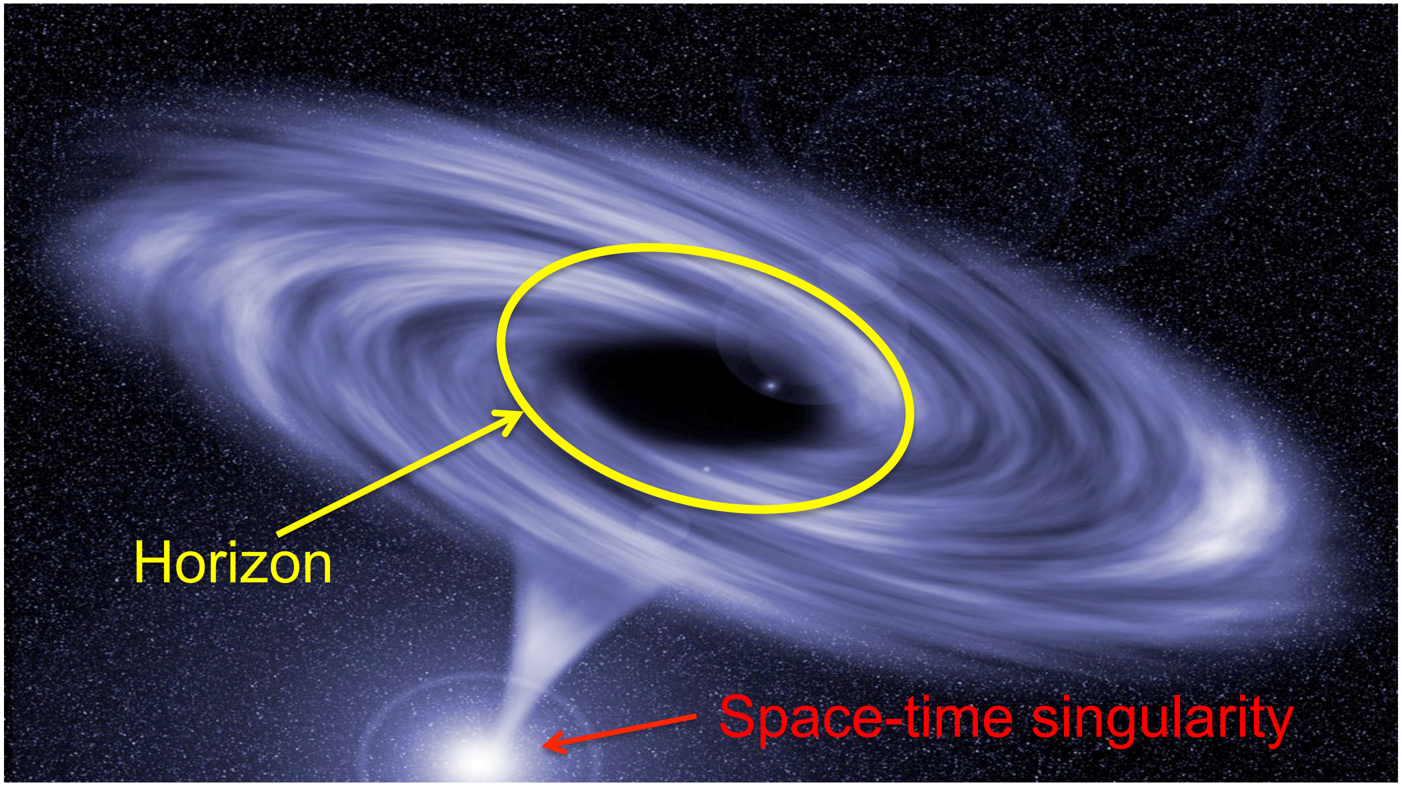
Bekenstein-Hawking - thermodynamic entropy: $S_{thermo}=\frac{1}{4} A_{hor}$

(A_{hor}= area of the black hole horizon; $c=\hbar=1;G_N=1$)

to

Statistical entropy: $S_{stat} = \log N_i$?

Where do black hole microscopic degrees N_i come from?



In String theory, main progress on microscopic origin of entropy primarily for extreme black holes w/

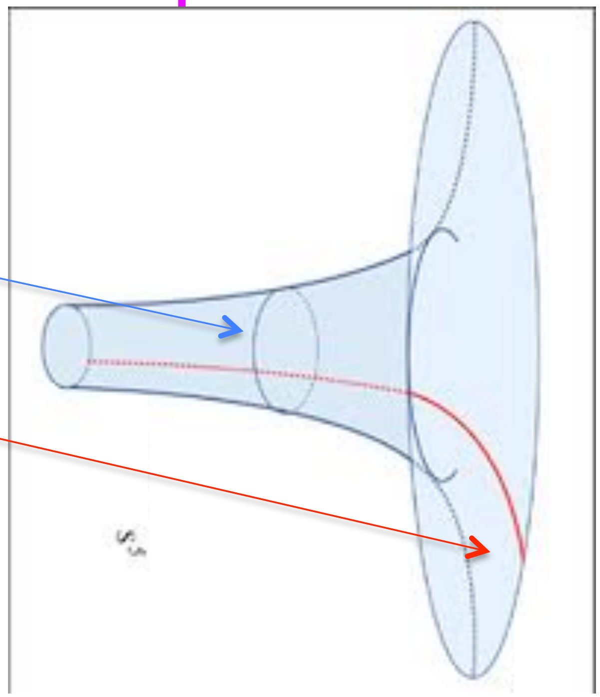
$$M = \Sigma_i |Q_i| + \Sigma_i |P_i| \qquad \text{(schematic)}$$

M-mass, Q_i-electric charges, P_i-magnetic charges
Extreme → primarily supersymmetric (BPS) multi-charged black holes, J=0 & w/ Λ=0.

Maldacena'97

Systematic study of microscopic degrees quantified via:
AdS_D/CFT_{D-1} (Gravity/Field Theory) correspondence

[A string theory on a specific Curved Space-Time (in D-dim.) related to specific Field Theory (in (D-1)-dim.) on its boundary]



Specific microscopic examples of black holes in string theory:
relation to 2-dim. CFT via AdS_3/CFT_2 correspondence
extensively explored:

- supersymmetric (BPS) multi-charged black holes w/ $\Lambda=0$:

  $M = \Sigma_i |Q_i| + \Sigma_i | P_i |$ $J=0$ Strominger, Vafa'96..

  (prototype: four-charge BPS dyon M.C., Youm 0507090)

- near-BPS I black holes:

  $M \sim \Sigma_i |Q_i| + \Sigma_i | P_i |$ Maldacena, Strominger'97..

- near-BPS multi-charged rotating black holes M.C., Larsen'98

Further developments:

- (near-)extreme rotating black holes

  $M > \Sigma_i |Q_i| + \Sigma_i | P_i |$, but $M^2 = J$

  Kerr/CFT correspondence Guica,Hartman,Song, Strominger'08

- extreme AdS charged rotating black holes in diverse dim.

  ….M.C., Chow, Lü, Pope 0812.2918

How about Microscopics of Non-extreme Black Holes?

$$M > \Sigma_i |Q_i| + \Sigma_i |P_i|$$

Motivation for Subtracting Geometry

Subtracted Geometry - Motivation

Introduced to quantify `past observations' (later) M.C., Larsen '97-'99
that not only extreme, but also non-extreme black holes might have microscopic explanation in terms of dual 2-dim. QFT: focus on non-extreme asymptotically flat (Λ=0), charged ,rotating black holes in four (and five) dimensions.

Should focus on the black hole `by itself' → enclose the black hole in a box (à la Gibbons Hawking), thus creating an equilibrium system with manifest conformal symmetry (later)

The box leads to a `mildly' modified geometry
changing only the warp factor $\Delta_0 \rightarrow \Delta$

⬇

Subtracted Geometry

M.C., Larsen 1106.3341,1112.4856

Non-extreme black holes in string theory w/ $\Lambda=0$

M - mass, Q_i, P_i - multi-charges, J - angular momentum

w/ $M > \Sigma_i |Q_i| + \Sigma_i |P_i|$

Prototype solutions of a sector of maximally supersymmetric D=4 Supergravity

[effective action of toroidally compactified string theory] → so-called STU model

STU Model Lagrangian

[A sector of toroidally compactified effective string theory]

$$2\kappa_4^2\mathcal{L}_4 = R\star 1 - \frac{1}{2}\star d\eta_a\wedge d\eta_a - \frac{1}{2}e^{2\eta_a}\star d\chi^a\wedge d\chi^a$$
$$-\frac{1}{2}e^{-\eta_0}\star F^0\wedge F^0 - \frac{1}{2}e^{2\eta_a-\eta_0}\star(F^a+\chi^a F^0)\wedge(F^a+\chi^a F^0)$$
$$+\frac{1}{2}C_{abc}\chi^a F^b\wedge F^c + \frac{1}{2}C_{abc}\chi^a\chi^b F^0\wedge F^c + \frac{1}{6}C_{abc}\chi^a\chi^b\chi^c F^0\wedge F^0$$

w/ A^0 & three gauge fields A^a, the three dilatons η^a and the three axions χ^a

Black holes: explicit solutions of equations of motion for the above Lagrangian w/ metric, four gauge potentials and three axio-dilatons

Prototype, four-charge rotating black hole, originally obtained via solution generating techniques

M.C., Youm 9603147
Chong, M.C., Lü, Pope 0411045

Four- SO(1,1) transfs.
time-reduced Kerr BH

$$H = \begin{pmatrix} \cosh\delta_i & \sinh\delta_i \\ \sinh\delta_i & \cosh\delta_i \end{pmatrix}$$

Full four-electric and four-magnetic charge solution only recently obtained

Chow, Compère 1310.1295;1404.2602

Metric of rotating four-charge black holes in String Theory:

M.C. & Youm 9603147
Chong, M.C., Lü & Pope 0411045

$$ds_4^2 = -\Delta_0^{-1/2} G(dt + \mathcal{A})^2 + \Delta_0^{1/2}\left(\frac{dr^2}{X} + d\theta^2 + \frac{X}{G}\sin^2\theta d\phi^2\right)$$

$X = r^2 - 2mr + a^2$ $= 0$ outer & inner horizon

$$G = r^2 - 2mr + a^2\cos^2\theta\ ,$$

$$\Pi_c \equiv \prod_{I=0}^{3}\cosh\delta_I\ , \quad \Pi_s \equiv \prod_{I=0}^{3}\sinh\delta_I\ .$$

$$\mathcal{A} = \frac{2ma\sin^2\theta}{G}\left[(\Pi_c - \Pi_s)r + 2m\Pi_s\right]d\phi\ ,$$

$$\Delta_0 = \prod_{I=0}^{3}(r + 2m\sinh^2\delta_I) + 2a^2\cos^2\theta[r^2 + mr\sum_{I=0}^{3}\sinh^2\delta_I + 4m^2(\Pi_c - \Pi_s)\Pi_s$$
$$- 2m^2\sum_{I<J<K}\sinh^2\delta_I\sinh^2\delta_J\sinh^2\delta_K] + a^4\cos^4\theta\ .$$

$$G_4 M = \frac{1}{4}m\sum_{I=0}^{3}\cosh 2\delta_I$$ Mass

$$G_4 Q_I = \frac{1}{4}m\sinh 2\delta_I \quad (I = 0,1,2,3)$$ Four charges

$$G_4 J = ma(\Pi_c - \Pi_s)$$ Angular momentum

Special cases:

$\delta_I = \delta$ Kerr-Newman

& a $= 0$ Reisner-Nordström;

$\delta_I = 0$ Kerr

& a $= 0$ Schwarzschild;

$\delta_I \to \infty$ $m \to 0$ w/m $\exp(2\delta_I)$-finite
extremal (BPS) black hole

Or equivalently : m, a, δ_I (I=0,1,2,3)

Thermodynamics - suggestive of weakly interacting 2-dim. CFT w/ `left-' & `right-moving' excitations

first noted, M.C., Youm'96
M.C., Larsen'97

Area of outer horizon $S_+ = S_L + S_R$

[Area of inner horizon $S_- = S_L - S_R$]

$$S_L = \pi\, m^2\,(\Pi_c + \Pi_s)$$

$$S_R = \pi\, m\sqrt{m^2 - a^2}\,(\Pi_c - \Pi_s)$$

Surface gravity (inverse temperature) of

outer horizon $\beta_H = ½\,(\beta_L + \beta_R)$

[inner horizon $\beta_- = ½\,(\beta_L - \beta_R)$]

$$\beta_L = 2\pi\, m\,(\Pi_c - \Pi_s)$$

$$\beta_R = \frac{2\pi\, m^2}{\sqrt{m^2 - a^2}}\,(\Pi_c + \Pi_s)$$

Similar structure for angular velocities Ω_+, Ω_- and momenta J_+, J_-.

Depend only on four parameters: m, a, $\Pi_c \equiv \prod_{I=0}^{3} \cosh\delta_I$, $\Pi_s \equiv \prod_{I=0}^{3} \sinh\delta_I$

Shown more recently, all independent of the warp factor Δ_0 !

M.C., Larsen'11

Determination of new warp factor $\Delta_0 \to \Delta$

via massless scalar field wave eq.: wave eq. separable &

the radial part is solved by hypergeometric functions w/ $SL(2,R)^2$

(manifest conformal symmetry→ promoted to 2-dim. CFT)

The general Laplacian (with warp factor Δ_0 implicit):

$$\Delta_0^{-\frac{1}{2}}[\partial_r X \partial_r - \frac{1}{X}(\mathcal{A}_{\rm red}\partial_t - \partial_\phi)^2 + \frac{\mathcal{A}_{\rm red}^2 - \Delta_0}{G}\partial_t^2 + \frac{1}{\sin\theta}\partial_\theta \sin\theta \partial_\theta + \frac{1}{\sin^2\theta}\partial_\phi^2]$$

w/ $\mathcal{A}_{\rm red} = \frac{G}{a\sin^2\theta}\mathcal{A} \; = 2m[(\Pi_c - \Pi_s)r + 2m\Pi_s]$

$\Delta_0 \to \Delta$ such that wave eq. is separable: f(r)+g(θ) (true for Δ_0 and Δ)

& the radial part is solved by hypergeometric functions:

uniquely fixed f(r)+g(θ)-const.

M.C., Larsen 1112.4856

Subtracted geometry for rotating four-charge black holes

$$ds_4^2 = -\Delta_0^{-1/2} G (dt + \mathcal{A})^2 + \Delta_0^{1/2} \left(\frac{dr^2}{X} + d\theta^2 + \frac{X}{G} \sin^2\theta d\phi^2 \right)$$

$$X = r^2 - 2mr + a^2 \ ,$$

$$G = r^2 - 2mr + a^2 \cos^2\theta \ ,$$

$$\mathcal{A} = \frac{2ma \sin^2\theta}{G} \left[(\Pi_c - \Pi_s) r + 2m\Pi_s \right] d\phi \ ,$$

$$\Delta_0 = \prod_{I=0}^{3} (r + 2m \sinh^2\delta_I) + 2a^2 \cos^2\theta [r^2 + mr \sum_{I=0}^{3} \sinh^2\delta_I + 4m^2 (\Pi_c - \Pi_s)\Pi_s - 2m^2 \sum_{I<J<K} \sinh^2\delta_I \sinh^2\delta_J \sinh^2\delta_K] + a^4 \cos^4\theta \ .$$

$$\boxed{\Delta_0 \to \Delta = (2m)^3 r (\Pi_c^2 - \Pi_s^2) + (2m)^4 \Pi_s^2 - (2m)^2 (\Pi_c - \Pi_s)^2 a^2 \cos^2\theta}$$

Comments: while $\Delta_0 \sim r^4$, $\Delta \sim r$

subtracted geometry depends only on four parameters:

m, a, $\Pi_c \equiv \prod_{I=0}^{3} \cosh\delta_I \ , \quad \Pi_s \equiv \prod_{I=0}^{3} \sinh\delta_I$

Remarks:

Asymptotic geometry of subtracted geometry is of a Lifshitz-type w/ a deficit angle:

$$ds^2 = -(\frac{R}{R_0})^{2p} dt^2 + B^2 dR^2 + R^2 (d\theta^2 + \sin^2\theta^2 d\phi^2)$$

p=3, B=4

→ black hole in an ``asymptotically conical box'' (w/ deficit angle)

M.C., Gibbons 1201.0601

→ the box is confining (``softer'' than AdS)

M.C., Gibbons 1201.0601

Matter fields (gauge potentials and scalars)

Scalars: $\eta_1 = \eta_2 = \eta_3 \equiv \eta,\ \chi_1 = \chi_2 = \chi_3 \equiv \chi,$

$$e^\eta = \frac{(2m)^2}{\sqrt{\Delta}}, \qquad \chi = \frac{a\,(\Pi_c - \Pi_s)}{2m}\cos\theta$$

Gauge potentials: $A^1 = A^2 = A^3 \equiv A.$

$$A^0 = \frac{(2m)^4 a\,(\Pi_c - \Pi_s)}{\Delta}\sin^2\theta d\phi + \frac{(2ma)^2\cos^2\theta\,(\Pi_c - \Pi_s)^2 + (2m)^4\Pi_c\Pi_s}{(\Pi_c^2 - \Pi_s^2)\,\Delta}dt,$$

$$A = \frac{2m\cos\theta}{\Delta}\left(\left[\Delta - (2ma)^2(\Pi_c - \Pi_s)^2\sin^2\theta\right]d\phi - 2ma\,(2m\Pi_s + r(\Pi_c - \Pi_s))\,dt\right),$$

Magnetic frame

w/ $\Delta = (2m)^3(\Pi_c^2 - \Pi_s^2)\bar{r} + (2m)^4\Pi_s^2 - (2ma)^2(\Pi_c - \Pi_s)^2\cos^2\theta$

M.C., Larsen 1112.4856

Lift of subtracted geometry on circle S^1 to five-dimension turns out to locally factorize $AdS_3 \times S^2$ ($[SL(2,R)^2 \times SO(3)]/Z_2$ symmetry)

[globally S^2 fibered over Bañados-Teitelboim-Zanelli (BTZ) black hole w/ mass M_3, angular momentum J_3 & 3d cosmol. const. $\Lambda=\ell^{-3}$]

$$ds_5^2 = \tfrac{2}{3}\left(ds_{S^2}^2 + ds_{BTZ}^2\right)$$

$$ds_{S^2}^2 = \tfrac{1}{4}\ell^2\left(d\theta^2 + \sin^2\theta d\bar{\phi}^2\right) \qquad \bar{\phi} = \phi + \frac{16ma(\Pi_c - \Pi_s)}{\ell^3}(z+t)$$

$$ds_{BTZ}^2 = -\frac{(r_3^2 - r_{3+}^2)(r_3^2 - r_{3-}^2)}{\ell^2 r_3^2}dt_3^2 + \frac{\ell^2 r_3^2}{(r_3^2 - r_{3+}^2)(r_3^2 - r_{3-}^2)}dr_3^2 + r_3^2\left(d\phi_3 + \frac{r_{3+}r_{3-}}{\ell r_3^2}dt_3\right)^2$$

$$\phi_3 = \frac{z}{R},$$

$$t_3 = \frac{\ell}{R}t,$$

$$r_3^2 = \frac{16(2mR)^2}{\ell^4}\left[2m(\Pi_c^2 - \Pi_s^2)r + (2m)^2\Pi_s^2 - a^2(\Pi_c - \Pi_s)^2\right]$$

Conformal symmetry of AdS_3 can be promoted to Virasoro algebra of dual two-dimensional CFT — à la Brown-Hennaux

Standard statistical entropy (via AdS_3/CFT_2) — à la Cardy

→ Reproduces entropy of 4D black holes

Further insights into geometry of subtracted geometry

i. Subtracted geometry as an infinite boost Harrison transformations on the original BH,SO(1,1) transformations [change aysmptotics]

$$H \sim \begin{pmatrix} 1 & 0 \\ \beta & 1 \end{pmatrix} \quad \beta \to 1$$

M.C. & Gibbons 1201.0601
Virmani 1203.5088
Sahay & Virmani 1305.2800..

ii. Interpolating geometries between the original black hole and their subtracted geometries obtained:

- via solution generating techniques

- dual CFT interpretation via deformation of irrelevant CFT operators

Baggio, de Boer, Jottar & Mayerson1210.7695
M.C., Guica & Saleem 1302.7032..

M.C., Larsen 1406.4536

Subtracted geometry [$\Delta_0 \rightarrow \Delta = A\, r + B \cos^2\theta + C$; A,B,C-horrendous]

also works for most general black holes of the STU-Model

(specified by mass, four electric and four magnetics charges, and angular momentum)

Chow & Compère 1310.1295;1404.2602

M.C. & Larsen 1106.3341

All also works in parallel for subtracted geometry of most general five-dimensional black holes

(specified by mass, three charges and two angular momenta)

M.C., Youm 9603100

Further recent developments

Quantum aspects of subtracted geometries:

i) Quasi-normal modes - exact results for scalar fields

two damped branches → no black hole bomb

M.C., Gibbons 1312.2250; M.C., Gibbons, Saleem 1401.0544

ii) Entanglement entropy - minimally coupled scalar

M.C., Satz, Saleem 1407.0310

iii) Vacuum polarization $<\varphi^2>$ analytic expressions

at the horizon: static M.C., Gibbons, Saleem, Satz 1411.4658

rotating M.C., Satz, Saleem 1506.07189

outside & inside horizon: rotating M.C., Satz 1605….

iv) Thermodynamics of subtracted geometry

via Komar integral: M.C., Gibbons, Saleem, 1412.5996 (PRL)

Thermodynamics via variational principle

An, M.C., Papadimitriou 1602.0150

- Identify normalizable and non-rormalizable modes:

Introduce new coordinates:

Rescaled radial coord.: $\ell^4 r \leftarrow (2m)^3(\Pi_c^2 - \Pi_s^2) r + (2m)^4 \Pi_s^2 - (2ma)^2(\Pi_c - \Pi_s)^2,$

Rescaled time: $\frac{k}{\ell^3} t \leftarrow \frac{1}{(2m)^3(\Pi_c^2 - \Pi_s^2)} t,$

Trade four parameters m, a, Π_c, Π_s for:

$$\ell^4 r_\pm = (2m)^3 m(\Pi_c^2 + \Pi_s^2) - (2ma)^2(\Pi_c - \Pi_s)^2 \pm \sqrt{m^2 - a^2}(2m)^3(\Pi_c^2 - \Pi_s^2)$$

$$\ell^3 \omega = 2ma(\Pi_c - \Pi_s), \qquad B = 2m,$$

r_+, r_-, ω - normalizable modes

B - non-renormalizable mode, along with ℓ and k
(fixed up to radial diffeomorphism and gl. Symmetries)

`Vacuum' solution

obtained by turning off r_+, r_-, ω – three normalizable modes:

$$ds^2 = \sqrt{r}\left(\ell^2 \frac{dr^2}{r^2} - rk^2 dt^2 + \ell^2 d\theta^2 + \ell^2 \sin^2\theta d\phi^2\right)$$

$$e^\eta = \frac{B^2/\ell^2}{\sqrt{r}}, \qquad \chi = 0, \qquad A^0 = 0, \qquad A = B\cos\theta d\phi$$

Non-normalizable (fourth) mode B, along with ℓ and k, fixed up to radial diffeomorphism:

$$r \to \lambda^{-4} r \qquad k \to \lambda^3 k, \quad \ell \to \lambda\ell, \quad B \to B$$

and global U(1) symmetry:

$$e^\eta \to \mu^2 e^\eta, \quad \chi \to \mu^{-2}\chi, \quad A^0 \to \mu^3 A^0, \quad A \to \mu A, \quad ds^2 \to ds^2$$

which keep kB^3/ℓ^3 - fixed

- # Radial Hamiltonian formalism

to determine S_{ct}, to the bulk action S

Suitable radial coordinate u, such that constant-u slices Σ_u

$$\Sigma_u \rightarrow \partial\mathcal{M} \quad \text{as } u \rightarrow \infty.$$

Decomposition of the metric and gauge fields:

$$ds^2 = (N^2 + N_i N^i)du^2 + 2N_i du dx^i + \gamma_{ij} dx^i dx^j$$
$$A^L = a^\Lambda du + A_i^\Lambda dx^i,$$

Decomposition leads to the radial Lagrangian L w/ canonical momenta:

$$\pi^{ij} = \frac{\delta L}{\delta \dot{\gamma}_{ij}}$$
$$\pi_I = \frac{\delta L}{\delta \dot{\varphi}^I}$$
$$\pi_\Lambda^i = \frac{\delta L}{\delta \dot{A}_i^\Lambda}$$

w/ momenta conjugate to N, N_i, and a_Λ vanish.

Hamiltonian:

$$H = \int \mathrm{d}^3\mathbf{x} \left(\pi^{ij}\dot{\gamma}_{ij} + \pi_I \dot{\varphi}^I + \pi^i_\Lambda \dot{A}^\Lambda_i \right) - L = \int \mathrm{d}^3\mathbf{x} \left(N\mathcal{H} + N_i \mathcal{H}^i + a^\Lambda \mathcal{F}_\Lambda \right)$$

First class constraints $\mathcal{H} = \mathcal{H}^i = \mathcal{F}_\Lambda = 0,$ - Hamilton Jacobi eqs.:

& Momenta as gradients of Hamilton's principal function $\mathcal{S}$(γ,A^,φ^I):

$$\pi^{ij} = \frac{\delta \mathcal{S}}{\delta \gamma_{ij}}, \quad \pi^i_\Lambda = \frac{\delta \mathcal{S}}{\delta A^\Lambda_i}, \quad \pi_I = \frac{\delta \mathcal{S}}{\delta \varphi^I}.$$

w/ original.

$$\pi^{ij} = \frac{\delta L}{\delta \dot{\gamma}_{ij}}$$

$$\pi_I = \frac{\delta L}{\delta \dot{\varphi}^I}$$

$$\pi^i_\Lambda = \frac{\delta L}{\delta \dot{A}^\Lambda_i}$$

deBoer,Verlinde^2'99,…Skenderis&Papadimitriou'04,…

Solve asymptotically (for `vacuum' asymptotic sol.) for

$\mathcal{S}$(γ,A^,φ^I) = - S_{ct} !

$\mathcal{S}$(γ,A^,φ^I) coincides with the on-shell action, up to terms that remain finite as $\Sigma_u \to \partial\mathcal{M}$. In particular, divergent part of $\mathcal{S}$[γ, A^, φ^I] coincides with that of the on-shell action.

`Hamiltonian Formalism with `Renonormalized' Action

$$S_{\text{reg}} = S_4 + S_{\text{ct}} \qquad S_{\text{ren}} = \lim_{r \to \infty} S_{\text{reg}}$$ Finite-independent of r

Covariant S_{ct} calculated for vacuum asymptotic sol.:

$$S_{\text{ct}} = -\frac{1}{\kappa_4^2} \int \mathrm{d}^3\mathbf{x}\sqrt{-\gamma}\, \frac{B}{4} e^{\eta/2} \left(\frac{4-\alpha}{B^2} + (\alpha - 1) e^{-\eta} R[\gamma] - \frac{\alpha}{2} e^{-2\eta} F_{ij} F^{ij} + \frac{1}{4} e^{-4\eta} F^0_{ij} F^{0ij} \right)$$

Renormalized canonical momenta:

$$\Pi^{ij} = \pi^{ij} + \frac{\delta S_{\text{ct}}}{\delta \gamma_{ij}}, \quad \Pi^i_\Lambda = \pi^i_\Lambda + \frac{\delta S_{\text{ct}}}{\delta A^\Lambda_i}, \quad \Pi_I = \pi_I + \frac{\delta S_{\text{ct}}}{\delta \varphi^I}$$

Conserved Charges:

Conserved currents, a consequence of the first class constraints

$F_\Lambda = 0$ Conserved currents for gauge potentials: $D_i\Pi^i = 0, \quad D_i\Pi^{0i} = 0.$

Conserved charges: $Q_4^{(m)} = -\int_{\partial\mathcal{M}\cap C} \mathrm{d}^2\mathbf{x}\,\Pi^t, \quad Q_4^{0(e)} = -\int_{\partial\mathcal{M}\cap C} \mathrm{d}^2\mathbf{x}\,\Pi^{0t}$

$= \frac{3B}{4G_4} \qquad = \frac{\ell^4}{4G_4B^3}\left(\sqrt{r_+r_-} + \omega^2\ell^2\right)$

$\mathcal{H}_i = 0$ Conserved currents: $-2D_j\Pi^j_i + \Pi_\eta\partial_i\eta + \Pi_\chi\partial_i\chi + F^0_{ij}\Pi^{0j} + F_{ij}\Pi^j \approx 0$

Conserved ``charges”: $\mathcal{Q}[\zeta] = \int_{\partial\mathcal{M}\cap C} \mathrm{d}^2\mathbf{x}\ \left(2\Pi^t_j + \Pi^{0t}A^0_j + \Pi^t A_j\right)\zeta^j$

Asymptotic Killing vector ζ_i

Mass: $M_4 = -\int_{\partial\mathcal{M}\cap C} \mathrm{d}^2\mathbf{x}\ \left(2\Pi^t{}_t + \Pi^t_0 A^0_t + \Pi^t A_t\right) = \frac{\ell k}{8G_4}(r_+ + r_-)$

Angular Momentum: $J_4 = \int_{\partial\mathcal{M}\cap C} \mathrm{d}^2\mathbf{x}\,(2\Pi^t{}_\phi + \Pi^t_0 A^0_\phi + \Pi^t A_\phi) = -\frac{\omega\ell^3}{2G_4}$

Thermodynamic relations and the first law

Free Energy: $I_4 = S^{\mathrm{E}}_{\mathrm{ren}} = -S_{\mathrm{ren}} = \beta_4 \mathcal{G}_4 = \dfrac{\beta_4 \ell k}{8G_4}\left((r_- - r_+) + 2\omega^2\ell^2\sqrt{\dfrac{r_-}{r_+}}\right)$

Quantum statistical relation: $\mathcal{G}_4 = M_4 - T_4 S_4 - \Omega_4 J_4 - \Phi^{0(e)} Q^{0(e)}$

First law: $\mathrm{d}M_4 - T_4 \mathrm{d}S_4 - \Omega_4 \mathrm{d}J_4 - \Phi_4^{0(e)}\mathrm{d}Q_4^{0(e)} - \Phi_4^{(m)}\mathrm{d}Q_4^{(m)} = 0.$

Smarr's Formula: $M_4 = 2S_4T_4 + 2\Omega_4 J_4 + Q_4^{0(e)}\Phi_4^{0(e)} + Q_4^{(m)}\Phi_4^{(m)}$

Varying parameters: r_+, r_-, ω, and B, k, ℓ subject to kB^3/ℓ^3 –fixed

⟷ original parameters m, a, Π_c, Π_s & a scaling parameter

Concluding Remarks

- Finite conserved charges and thermodynamic identities are a consequence of a well posed variational problem, formulated in terms of equivalence classes of boundary data under the asymptotic local symmetries. Demonstrated this for asymptotically conical backgrounds of the STU model.

- Uplifting these solutions to five dimensions, give a precise map between all thermodynamic variables of subtracted geometries and those of the BTZ black hole. Some free parameters (B, ω) of the four-dimensional black holes are fixed (B) or quantized (ω) in order for the solutions to be uplifted to five dimensions.

- Analysis does not assume or imply holographic duality for asymptotically conical backgrounds, but is a first step in this direction.

Work in progress