Black Holes in Higher-Derivative Gravity

Classical and Quantum Black Holes

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Introduction

- Black holes are the most fundamental objects in any theory of gravity, and as such they can provide powerful probes for investigating some of the subtle global features of the theory.

- In ordinary Einstein gravity, by itself or coupled to "standard" matter, there are powerful no-hair theorems or uniqueness theorems which imply strong restrictions on the parameter space of possible black-hole solutions. For example, in Einstein-Maxwell theory the most general black hole is characterised by its mass, angular momentum and charge.

- More general theories of gravity comprise Einstein gravity with higher-order curvature terms, such as arise in the low-energy limit of string theory. In string theory, there are an infinite number of higher-order terms, involving arbitrarily large powers of the curvature and its covariant derivatives.

- We may first consider a theory with just a finite number of higher-order terms. Of particular interest is the case of Einstein gravity with additional quadratic curvature terms only, since this is actually renormalisable (Stelle 1977), albeit at the price of having ghosts.

- One may perhaps find a regime where quadratic curvatures dominate over yet higher order terms (Starobinsky,...). In string theory, maybe there are regions in the *String Theory Landscape* where the quadratic curvature terms are dominant (Brigante, Hong Liu, Myers, Shenker, Yaida).

- In any case, it is worthwhile to investigate exactly what are the black-hole solutions in quadratic-curvature modified gravity, since it is a tractable problem. (In more than four dimensions, for example, Einstein with a Gauss-Bonnet term admits exact solutions for black holes (Deser & Boulware).)

- We consider spherically-symmetric black holes. In four dimensions, unlike higher dimensions, any solution of of Einstein gravity remains a solution when quadratic curvature terms are added. Thus the Schwarzschild black hole is a solution in the quadratic theory. The question is whether there exist any other acceptable spherically-symmetric black-hole solutions.

- For example, can there exist non-standard black holes in a regime where cubic and higher terms can be neglected? If so, these solutions would be representative of new solutions even in string theory.

- As we shall see, there in fact exists a second branch of such black holes in Einstein plus quadratic gravity. This branch intersects the usual Schwarzschild branch, and in the cross-over region, can be studied perturbatively.

- The new features of the higher-derivative theory are associated with additional modes in the spectrum of the theory; the massive spin-2 and spin-0 modes. The new black holes involve condensates of the (ghost-like) massive spin-2 modes.

General Expectations in Quadratic Gravity

- Since the Gauss-Bonnet combination of quadratic curvatures is a total derivative in four dimensions, the most general action for the quadratic theory (without cosmological constant) can be taken to be

$$I = \int d^4x\, \sqrt{-g}\, (R - \alpha\, C_{\mu\nu\rho\sigma} C^{\mu\nu\rho\sigma} + \beta\, R^2)\,.$$

  If one linearises the theory around a Minkoswki background, the fluctuations will obey equations which can be separated into a factorised fourth-order spin-2 equation and a second-order spin-0 equation:

$$(\Box - m_2^2)\, \Box\, h_{\mu\nu} = 0\,, \qquad (\Box - m_0^2)\phi = 0\,,$$

  where $m_2^2 = 1/(2\alpha)$ and $m_0^2 = 1/(6\beta)$.

- The massive spin-2 and spin-0 modes will lead to terms with Yukawa-type behaviour $1/r\, e^{\pm mr}$ at large distances. Generically, terms with both signs in the exponential will occur, and the terms with the rising exponentials will give rise to fatal pathological behaviour in the asymptotic region.

- The real question, therefore, is whether it is possible to fine tune the parameters in the general solutions so as to be able to remove the rising Yukawa terms.

- Are there any such fine tunings, aside from Schwarzschild?

A Trace No-Hair Theorem

- We can study the static solutions of the theory by considering metrics of the form

$$ds^2 = -\lambda^2\, dt^2 + h_{ij} dx^i dx^j \,,$$

where λ and h_{ij} depend only on the spatial coordinates x^i.

- The equations of motion for the quadratic theory are

$$R_{\mu\nu} - \tfrac{1}{2} R g_{\mu\nu} - 4\alpha\, B_{\mu\nu} + 2\beta\, R(R_{\mu\nu} - \tfrac{1}{4} R g_{\mu\nu}) + 2\beta (g_{\mu\nu}\Box - \nabla_\mu \nabla_\nu) R = 0\,,$$

where $B_{\mu\nu} = (\nabla^\rho \nabla^\sigma + \frac{1}{2} R^{\rho\sigma}) C_{\mu\rho\nu\sigma}$ is the Bach tensor. Taking the trace gives

$$\beta\,(\Box - m_0^2) R = 0\,.$$

- Multiplying by λR and integrating over the spatial domain outside the putative horizon of the black hole gives

$$\int d^3x \sqrt{h}\, \Big[D^i(\lambda R D_i R) - \lambda (D_i R)^2 - m_0^2\, \lambda R^2 \Big] = 0\,,$$

where D_i are covariant derivatives in the spatial metric h_{ij}. Since by definition λ vanishes on the horizon, it follows that if $D_i R$ goes to zero sufficently rapidly at infinity then the surface term gives no contribution and hence the non-positivity of the remaining integrand implies

$$R = 0\,.$$

- This partial no-hair theorem (due to W. Nelson) provides a considerable simplification of the problem. It means we immediately have a second-order equation of motion, and in fact if we use the spherically-symmetric ansatz

$$ds^2 = -h(r)\, dt^2 + \frac{dr^2}{f(r)} + r^2(d\theta^2 + \sin^2\theta\, d\phi^2)\,,$$

 we can reduce the equations of motion to two second-order ODEs for $h(r)$ and $f(r)$.

- Nelson tried to go further, by applying a similar technique to the trace-free part of the equations of motion, and claiming this led to a complete no-hair theorem, namely that $R_{\mu\nu} = 0$. This would have meant that the Schwarzschild metric was the only spherically-symmetric static black hole solution in the quadratic theory.

- However, we found that Nelson had some crucial sign errors in his calculation, and what he had thought to be a strictly non-positive integrand actually had a mix of terms with both signs. Thus no definite conclusion could be reached.

- On the basis that "what is not forbidden is allowed," this raises the possibility of non-Schwarzschild spherically-symmetric static black holes in the quadratic theory. With the simplification of knowing that they must satisfy $R = 0$, this means we can wolog drop the R^2 term in the action and just consider Einstein-Weyl gravity.

Numerical Solution of the Equations

- The $R = 0$ condition enables us to reduce the equations for h and f to two coupled non-linear 2nd-order ODEs. Unfortunately these appear not to be explcitly solvable, and we therefore resort to numerical analysis.

- We begin by making Taylor expansions of $h(r)$ and $f(r)$ near to a putative horizon at $r = r_0$:

$$\begin{aligned} h &= c\Big[(r - r_0) + h_2\,(r - r_0)^2 + h_3\,(r - r_0)^3 + \cdots\Big]\,, \\ f &= f_1\,(r - r_0) + f_2\,(r - r_0)^2 + f_3\,(r - r_0)^3 + \cdots\,. \end{aligned}$$

The constant c is a trivial, absorbable into a rescaling of the time coordinate. Plugging into the equations of motion, one can solve for all the coefficients h_i and f_i with $i \ge 2$ in terms of the two non-trivial parameters r_0 and f_1. The Schwarzschild solution itself corresponds to $f_1 = 1/r_0$, so if we write

$$f_1 = \frac{1 + \delta}{r_0}\,,$$

then non-vanishing δ characterises the extent to which the near-horizon solution deviates from Schwarzschild.

- We can now use the shooting method to construct solutions numerically. Namely, we set initial conditions just outside the horizon, by choosing values for r_0 and δ and making use of the near-horizon Taylor expansions. We then integrate the equations out numerically to large r.

- It is convenient to fix the scale size in the problem by making a choice for the coupling constant α for the Weyl-squared term in the action. We take $\alpha = \frac{1}{2}$. The parameters r_0 and δ are then both non-trivial.

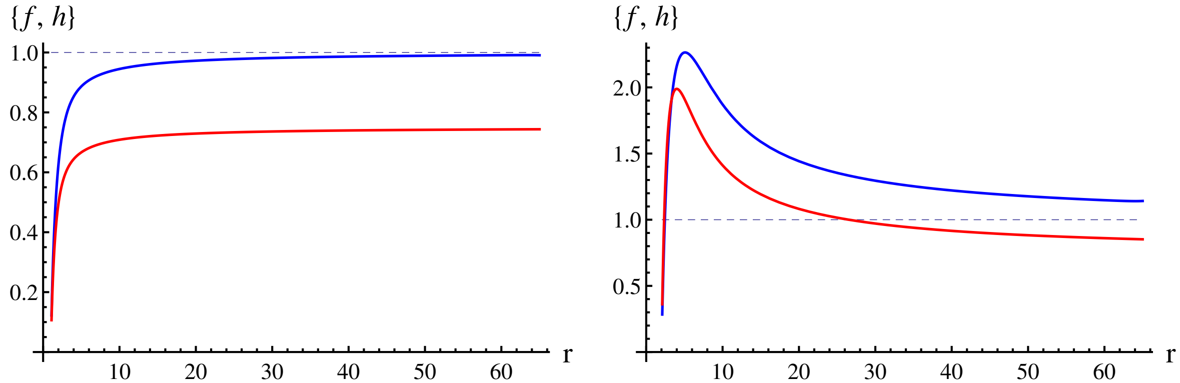
- For generic r_0 and δ, the outward integration runs into a singularity, in which the metric functions $h(r)$ and $f(r)$ rapidly diverge either to $+\infty$ or $-\infty$. By very delicate fine tuning of the parameters, one can systematically extend outwards the limit $r_{\rm max}$ before which the singularity is reached. Increasing the precision allows integrating out further–ad infinitum.

- In the non-singular fine-tuned solutions, $h(r)$ and $f(r)$ asymptotically approach constants, with $f(r) \to 1$. The asymptotic constant value for $h(r)$ can be adjusted by choosing the trivial parameter c so that $h(\infty) = 1$. Thus we obtain well behaved asymptotically flat black hole solutions.

- Procedure: Pick a value for r_0, and then fine tune δ to get asymptotically Minkowskian behaviour. For any r_0 there is always at least one such solution, with $\delta = 0$ (to within numerical rounding errors), corresponding to the Schwarzschild black hole.

- In addition, if r_0 is greater than a certain minimum value $r_0^{\text{min}} \approx 0.876$, we found that there exists a second choice of $\delta = \delta^*$ that gives a second, non-Schwarzschild, black hole.

- Here are two examples, showing the $f(r)$ (blue) and $h(r)$ (red) metric functions for the non-Schwarzschild black hole, for $r_0 = 1$ (LHS), and $r_0 = 2$ (RHS). In order to avoid an asymptotic overlay of the h and f curves, we have chosen the trivial scaling c so that the function h asymptotically approaches $\frac{3}{4}$ rather than 1.

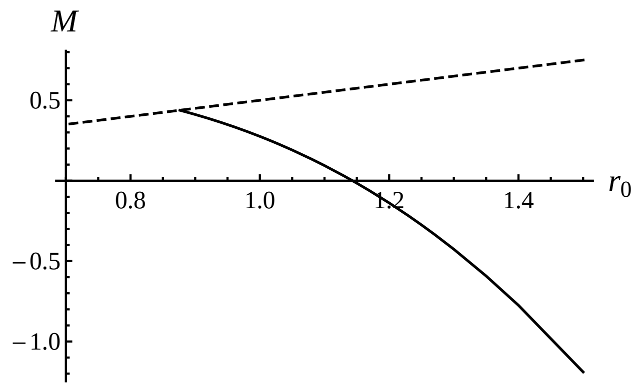


- The metric functions in the $r_0 = 1$ case are looking very like those in Schwarzschild. In the $r_0 = 2$ case they aproach their asymptotic values from above, suggesting negative mass.

Properties of the Non-Schwarzschild Black Holes

- The mass of the non-Schwarzschild black hole is given by the usual ADM formula for asymptotically flat spacetimes, which amounts to $\frac{1}{2}$ the coefficient of the $1/r$ term in g_{tt} (assuming a canonical normalisation for t so that $g_{tt} \to -1$ at infinity).

- The mass of the Schwarzschild (dotted line) and the non-Schwarzschild (solid line) black holes as a function of horizon radius r_0 are shown below:



- We originally thought the solid curve for the non-Schwarzschild black hole terminated at the left-hand end at $r_0 = r_0^{\text{min}}$. The non-Schwarzschild and Schwarzschild branches coalesce here, at r_0.

- In a further, recent, analysis, we find that the non-Schwarzschild branch in fact continues through to small r_0 values also. (The previous numerical analysis became delicate at $r_0 = R_{\text{min}}$.)

Negative Mass?

- The negative mass of the non-Schwarzschild black holes for $r_0 > r_0^{m=0}$ clearly violates the normal behaviour seen for black holes in general relativity, for which the positive energy theorem guarantees the non-negativity of the mass for any system of physically reasonable matter coupled to gravity.

- In fact the behaviour we are seeing here is very like what would happen if we looked at ordinary Schwarzschild black holes in pure Einstein gravity, but with a minus sign in front of the Einstein-Hilbert action. The mass, calculated as a Noether charge for the sign-reversed action, would be negative, and it would become more and more negative as the radius r_0 of the black hole was increased.

- The negativity of the mass for large non-Schwarzschild black holes can be understood as being a consequence of the ghost-like nature of the massive spin-2 modes in quadratic gravity:

$$\frac{m_2^2}{\Box\,(\Box - m_2^2)} = \frac{1}{\Box} - \frac{1}{(\Box - m_2^2)}\,.$$

  Effectively, we are seeing that whereas a condensation of massless spin-2 gravitons in a normal black hole gives rise to a spacetime with positive energy, a condensation of massive spin-2 modes, which are ghostlike, can give rise to a spacetime with negative energy.

Linearisation Around Schwarzschild

- The non-Schwarzschild black holes form a distinct branch that only intersects the Schwarzschild branch at $r_0 = r_0^{\text{min}}$. They have positive mass only when
$$r_0 \le r_0^{m=0} \approx 1.143\,.$$
- For r_0 close to r_0^{min}, the non-Schwarzschild black hole is perturbatively close to the Schwarzschild black hole. Apart from this case, the non-Schwarzschild black holes cannot in general be obtained by a linearised analysis around Schwarzschild.
- If we look for black hole solutions in a linearised analysis around Schwarzschild, then $R_{\mu\nu} = 0$ in the Schwarzschild background, and by the trace no-hair theorem the varied Ricci scalar is zero, $\delta R = 0$. Then $g^{\mu\nu}\,\delta R_{\mu\nu} = 0$, and the linearised Bianchi identity implies $\nabla^\mu\,\delta R_{\mu\nu} = 0$. Thus the linearised equations of motion imply that $\delta R_{\mu\nu}$ is transverse and traceless, obeying
$$(\Delta_L + m_2^2)\,\delta R_{\mu\nu} = 0\,,$$
where Δ_L is the Lichnerowicz operator,
$$\Delta_L\,\delta R_{\mu\nu} = -\Box\,\delta R_{\mu\nu} - 2R_{\mu\rho\nu\sigma}\,\delta R^{\rho\sigma}$$
in the Schwarzschild background. For $m_2^2 > 0$, this requires that Δ_L have a negative-eigenvalue mode.
- As first discussed by Gross, Perry, Yaffe (1982), Schwarzschild has just one normalisable negative-eigenvalue TT mode, with $\lambda \approx -0.7677\,r_0^{-2}$.

- In our numerical analysis we set $\alpha = \frac{1}{2}$, which implies the massive spin-2 field has $m_2^2 = 1/(2\alpha) = 1$. Thus for the solution linearised around Schwarzschild we require $\lambda = -m_2^2$, and hence $0.7677 r_0^{-2} \approx 1$, which gives $r_0 \approx \sqrt{0.7677} \approx 0.876$. This indeed reproduces the r_0^{min} horizon radius for which our numerical analysis indicated the bifurcation of the non-Schwarzschild branch of black holes. (The possibility of this bifurcation was foreseen by Brian Whitt (1985).)

- Analogous non-Schwarzschild spherically-symmetric black holes can arise in higher dimensions. In dimension $n \ge 5$ there are three independent quadratic-curvature invariants; $R^{\mu\nu\rho\sigma} R_{\mu\nu\rho\sigma}$, $R^{\mu\nu} R_{\mu\nu}$ and R^2. (The Gauss-Bonnet combination is no longer a total derivative in $n \ge 5$.) Only the theories without the $R^{\mu\nu\rho\sigma} R_{\mu\nu\rho\sigma}$ term continue to admit Schwarzschild as a solution, so for a linearised analysis we must restrict attention to the subset of theories:

$$I = \int d^n x \sqrt{-g} (R + c_1 R^{\mu\nu} R_{\mu\nu} + c_2 R^2) \,.$$

- There is a normalisable negative-eigenvalue Lichnerowicz TT mode in each dimension, and so a similar phenomenon of a bifurcating non-Schwarzschild branch arises. However, since the trace of the field equation now has quadratic-curvature terms,

$$(1 - \tfrac{1}{2}n)(R + c_1 \Box R) + (c_1 + 2c_2)(n-1)\Box R + 2(1 - \tfrac{1}{4}n)(c_1 R^{\mu\nu} R_{\mu\nu} + c_2 R^2) = 0 \,,$$

  we no longer have a trace no-hair theorem $R = 0$ for spherically-symmetric black holes, and so a numerical analysis moving out from the bifurcation along the non-Schwarzschild branch is much more complicated. (But we *can* still prove $\delta R = 0$ for perturbations away from Schwarzschild, which is why a negative Lichnerowicz TT mode again signals a bifurcation.)

Time Dependent Stability?

- One can now study time-dependent perturbations around Schwarzschild, so look for instabilities. The perturbations in $\delta R_{\mu\nu}$ are again TT, and satisfy $(\Delta_L + m_2^2)\,\delta R_{\mu\nu} = 0$. Since $\delta R_{\mu\nu} = \frac{1}{2}\Delta_L \delta g_{\mu\nu}$, the metric perturbations will likewise be TT and satisfy $(\Delta_L + m_2^2)\,\delta g_{\mu\nu} = 0$.

- The spherically-symmetric time-dependent TT perturbations take the form

$$\psi_{\mu\nu} dx^\mu dx^\nu = e^{-\mathrm{i}\,\omega t}\left[h(r)\,\psi_0(r)\,dt^2 + h^{-1}(r)\,\psi_1(r)\,dr^2 + 2\chi(r)\,dtdr + r^2\,\bar{\psi}(r)\,d\Omega_{n-2}^2\right]$$

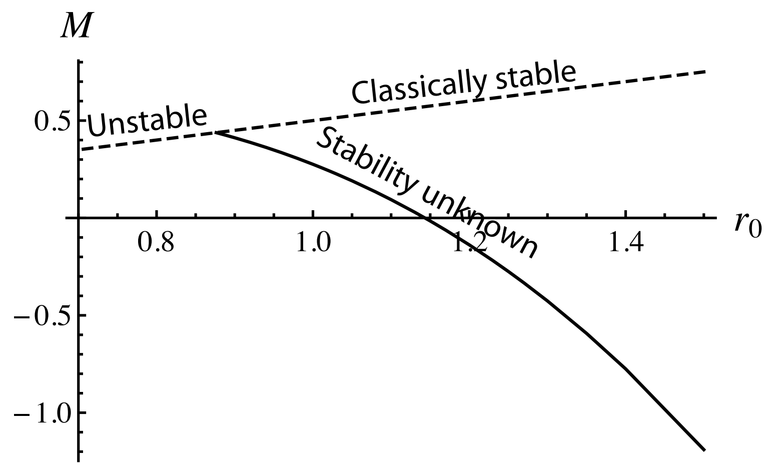
  where $h(r) = 1 - r^{3-n}$ is the n-dimensional Schwarzschild metric function. The equations can be boiled down to a second-order ODE for $\psi_1(r)$.

- We are interested in knowing when there exist such time-dependent solutions for which ω has a positive imaginary part, signalling run-away exponential time dependence. This same question was addressed by Gregory & LaFlamme (1993), in the context of five-dimensional black strings, $ds_5^2 = ds_4^2 + dz^2$, where ds_4^2 is the Schwarzschild metric. For z dependence $e^{\mathrm{i}\,kz}$, the problem reduces to studying time-dependent TT modes in four dimensions, satisfying $(\Delta_L + k^2)\psi_{\mu\nu} = 0$.

- If $\bar{\lambda} = -0.7677 r_0^{-2}$ denotes the Gross-Perry-Yaffe negative eigenvalue of the static mode, it turns out that time-dependent modes with an imaginary part to ω occur for Lichnerowicz eigenvalues $\bar{\lambda} < \Delta_L < 0$. In our context, this means that a runaway instability will occur for Schwarzschild black holes whose radius is less than the critical value at the bifurcation point.

Time Dependent Stability?

- The situation regarding the classical stability of the Schwarzschild branch of black holes is thus understood. The stability, or otherwise, of the non-Schwarzschild branch requires further investigation:



Asymptotically AdS Black Holes

- Adding a cosmological term $-2\Lambda\sqrt{-g}$ to the action gives rise (for negative Λ) to asymptotically AdS solutions. The trace no-hair result now shows that black holes solutions must have $R = 4\Lambda$, which again greatly simplifies the equations. Again, we get two second-order ODEs, which can only be solved numerically (apart from Schwarzschild-AdS, which is an exact solution).

- The numerical "shooting" method now becomes rather easy, since instead of $1/r\, e^{\pm m_2 r}$ asymptotics, the two possible asymptotic behaviours are both inverse power-law $1/r^{a_\pm}$, i.e. $a_+ > a_- > 0$. (For appropriate range of massive spin-2 mass m_2.)

- A similar situation occurs with simpler systems such as two-derivative Einstein-Scalar or Einstein-Proca theories. When $\Lambda = 0$, there in fact exist no-hair theorems showing that the black black-hole solutions cannot carry any scalar or Proca "hair." But when $\Lambda < 0$, the asymptotic fall-offs for the scalar or Proca field are such that solutions with non-vanishing scalar or Proca hair can easily arise. (Can only solve numerically.)

Conclusions

- Even though Einstein plus quadratic gravity generically has rising as well as falling Yukawa terms in the asymptotic solutions, one can fine tune the parameters in static spherically symmetric solutions and thereby find a second branch of solutions, over and above Schwarzschild. They have positive mass for $r_0 < 1.143\sqrt{2\alpha}$, and they are perturbatively close to Schwarzschid for $r_0 \approx 0.876\sqrt{2\alpha}$.

- Knowing what the black hole solutions in the theory are numerically allows us to see what one should or should not try to prove analytically.

- For example, we now know that one should not try to prove in general that spherically-symmetric non-Schwarzschild black holes can't exist in Einstein + quadratic gravity.

- It also shows that black holes can exist that are not perturbatively close to Schwarzschild.

- Interesting questions remain, including:

  Stability of the non-Schwarzschild black holes?

  Can one live with the ghosts/phantoms/poltergeists of higher-derivative gravity?

  Do these black holes have any relevance in string theory, where there are higher-order curvature corrections to arbitrarily high order?

  Might there exist no-hair theorems even in string theory?