Capacity of Entanglement and Quantum Hypothesis testing

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Based on:
JdB, Jarkko Järvelä,
Esko Keski-Vakkuri arXiv:1807.07357
JdB, Victor Godet, Jani Kastikainen,
Esko Keski-Vakkuri, arXiv:2006.nnnnn

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Outline:

1. Capacity of Entanglement
2. Properties
3. Area Law
4. Relative Entropy Variance
5. Quantum Hypothesis Testing
6. Optimal Measurements
7. Conclusion

Capacity of Entanglement

Consider a density matrix ρ with eigenvalues $\lambda_i = e^{-E_i}$

This set is sometimes called the entanglement spectrum. One can also interpret the system as a thermal system at unit temperature with energies E_i and Hamiltonian $K = -\log\rho$

Entanglement entropy:

$$S = -\mathrm{Tr}(\rho\log\rho) = -\sum_i \lambda_i \log\lambda_i = \sum_i E_i e^{-E_i} = \langle K\rangle$$

Capacity of entanglement:

$$C = \mathrm{Tr}(\rho\log^2\rho) - (\mathrm{Tr}(\rho\log\rho))^2 = \sum_i E_i^2 e^{-E_i} - \left(\sum_i E_i e^{-E_i}\right)^2$$
$$= \langle K^2\rangle - \langle K\rangle^2$$

Yao, Qi ’10; Schliemann ‘10

Capacity of entanglement

- Is the direct analogue of heat capacity
- Is related to gravitational fluctuations in the bulk
- May be connected to actual energy spectrum of parent theory and perhaps phase structure (*)
- Is a direct diagnostic of the deviation from a flat spectrum (C=0 for EPR pairs)
- Appears to have universal features such as an area law in CFTs

* Chandran, Khemani, Sondhi ‘14
Nakagawa, Furukawa ‘17

Relation to Renyi entropy:

$$C = \partial_\alpha^2 \log \mathrm{Tr}\rho^\alpha|_{\alpha=1}$$

Bound (cf Popoviciu: $\Delta X^2 \leq \frac{1}{4}(M-m)^2$)

$$C \leq \frac{1}{4} S_{\max}^2$$

We will see that in many case C~S with an area law for C if there is an area law for S.

What is the fundamental meaning of this (if any)?

- Simple example $\rho(E) = \frac{E^k}{k!}$ has C=S

- 2d CFT: $K = 2\pi \int_{|x|<R} dx \frac{R^2 - x^2}{2R} T_{tt}(x)$

$$C = \langle K^2 \rangle_c = \int_{-R}^{R} dx \int_{-R}^{R} dx' \frac{(R^2 - x^2)(R^2 - x'^2)}{4R^2} \frac{c}{(x - x')^4}$$

$$\sim \frac{c}{3} \log \frac{L}{a} = S$$

Hung, Myers, Smolkin '14

Entanglement spectrum in holographic CFTs

Hung, Myers, Smolkin, Yale ‘11

$$\int_0^\infty dE \rho(E) e^{-\alpha E} = e^{-f(\alpha)}$$

$$f(\alpha) = \alpha S_{EE} \left(1 - \frac{x_\alpha^d}{2} - \frac{x_\alpha^{d-2}}{2}\right)$$

$$x_\alpha = \frac{1 + \sqrt{1 + \alpha^2 d(d-2)}}{\alpha d}$$

$$\alpha \to \infty, \qquad f(\alpha) \sim E_c \alpha$$

$$E_c = S_{EE} \left(1 - \left(\frac{d-2}{d}\right)^{\frac{d}{2}} - \left(\frac{d-2}{d}\right)^{\frac{d}{2}-1}\right)$$

for a spherical region in vacuum only

density of states takes universal form:

$$\log \rho(E) \sim E_c^{1/d}(E - E_c)^{(d-1)/d}$$

2d:
Calabrese, Lefevre '10
Alba, Calabrese, Tonni '17

One might expect universal high/low temperature behavior but not clear why we have

$$C = -f''(1) = S = f'(1)$$

To get an area law we need the weaker condition

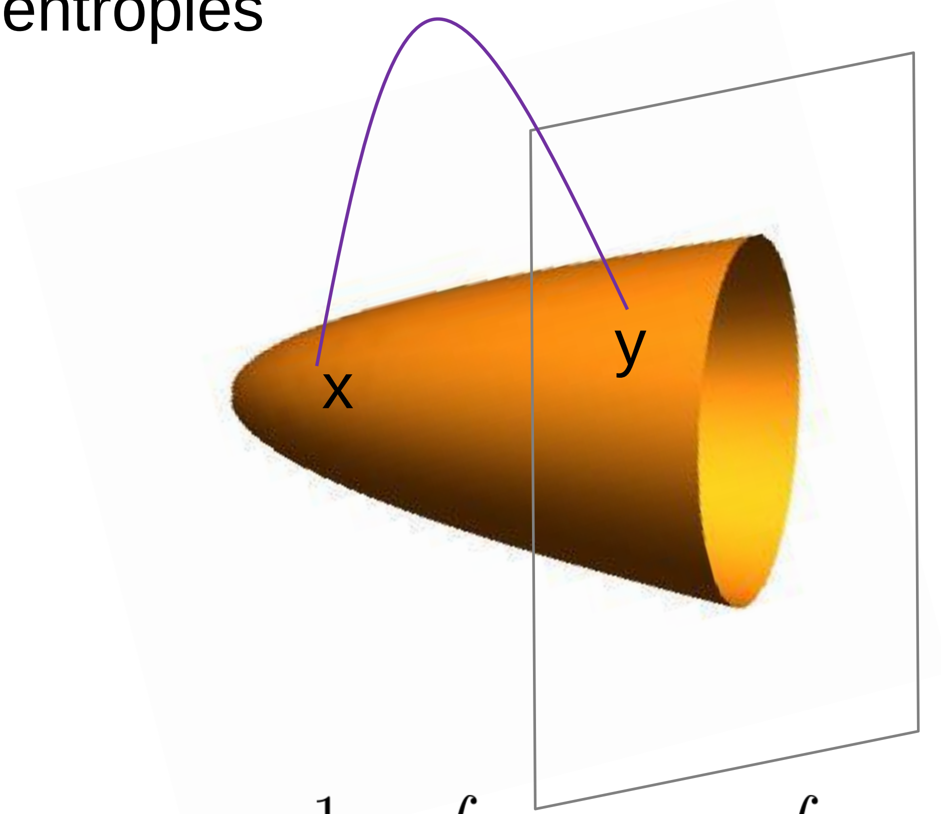
$$\log \rho(E) \sim \frac{1}{a_{\mathrm{UV}}^{d-2}} \log \hat{\rho}(E a_{\mathrm{UV}}^{d-2}) + \dots$$

Seems reasonable from a local Rindler point of view and might be universal in QFT

Holographic dual

Nakaguchi, Nishioka, '16

Use eg Xi Dong ('16) description of brane dual of Renyi entropies



cf Landau-Ginzburg:

$$\chi = \int dx dy \langle \vec{m}(x) \cdot \vec{m}(y) \rangle$$

$$C = \frac{1}{64G^2} \int dx \sqrt{h(x)} \int dy \sqrt{h(y)} h^{ij} G_{ij;kl}(x,y) h^{kl}$$

Here, area law arises from behavior when x,y approach the boundary of AdS. So C is not a good probe of bulk graviton fluctuations…

In systems with a conserved U(1) charge that can be mapped to free fermion systems, one has a relation between particle number fluctuations and S,C

Klich et al ‘06-’11
Calabrese, Mintchev, Vicari ‘11

$$\begin{aligned} S &= \frac{\pi^2}{3} n_2 + \frac{\pi^4}{45} n_4 + \dots \\ C &= \frac{\pi^2}{3} n_2 + \frac{8\pi^4}{45} n_4 + \dots \end{aligned}$$

- For Gaussian particle fluctuations, S=C
- For large N, can also show S=C
- Lesson for large N holographic theories?

Random pure states in $\mathcal{H}_p \otimes \mathcal{H}_q$ Lloyd, Pagels '88

For p=q=2, C/S~1.08

For p=2, q→∞ C/S→0

For p=q→∞, C/S→0

This suggests that when C and S are approximately equal, most of the entanglement is carried by randomly entangled pairs of qubits (not EPR pairs)… relevant for quasiparticle pictures of entanglement generation in quenches.

More comments on C/S

- Ratio is scheme dependent: tested with free massless scalar in d=3. Using mode sums (Srednicki '93) get C/S~3, using heat kernel (Solodukhin '11) get C/S=1.

- Gauss-Bonnet in 5d, C/S=c/a (in natural scheme)

- C/S changes under perturbative shape deformations

- C/S changes under relevant deformations

Upshot: capacity of entanglement does seem to have a universal area law in QFT, C/S is a scheme and model dependent ratio of order unity which contains interesting quantitative and qualitative information.

To have a better probe of fluctuations, should perhaps consider fluctuations associated to UV finite quantities such as relative entropy and mutual information.

One such quantity is relative entropy variance (also known as quantum relative variance or quantum information variance)

$$V(\rho|\sigma) = \mathrm{Tr}[\rho(\log\rho - \log\sigma)^2] - (\mathrm{Tr}[\rho(\log\rho - \log\sigma)])^2$$

Can this quantity be computed and does is contain interesting physics information?

Does this quantity have an operational meaning?

Quantum hypothesis testing

Given two states ρ, σ and a POVM $(E_0, E_1) = (A, 1 - A)$ we can make a measurement in both states and identify the state depending on whether the outcome is “0” or “1”. The probabilities to make mistakes are

$$\begin{aligned} \alpha &= \mathrm{Tr}(\rho(1 - A)) && \text{Type I} \\ \beta &= \mathrm{Tr}(\sigma A) && \text{Type II} \end{aligned}$$

Of particular interest is measurements on n copies of the system in which case

$$\begin{aligned} \alpha_n &= \mathrm{Tr}(\rho^{\otimes n}(1 - A^{(n)})) && \text{Type I} \\ \beta_n &= \mathrm{Tr}(\sigma^{\otimes n} A^{(n)}) && \text{Type II} \end{aligned}$$

Symmetric Testing

Minimize the sum of the errors over all possible measurements

$$P_n^* = \frac{1}{2} \inf_{A^{(n)}} \mathrm{Tr}\left(\rho^{\otimes n}(1 - A^{(n)}) + \sigma^{\otimes n} A^{(n)}\right)$$

Quantum Chernoff bound (Audenaert et al ’07)

$$\lim_{n\to\infty}\left(-\frac{1}{n}\log P_n^*\right) = -\log Q(\rho,\sigma) = \max_{0\leq s\leq 1}[-\log \mathrm{Tr}\rho^s\sigma^{1-s}]$$

Optimal strategy: $A^{(n)}$ is the projector on the non-negative eigenvalue subspace of $\rho^{\otimes n} - \sigma^{\otimes n}$

Asymmetric Testing: optimize with respect to an a priori given bound on the type I error

$$\beta_n^*(\varepsilon) \equiv \inf_{A^{(n)}} \{\beta_n \mid \alpha_n \leq \varepsilon\}$$

Asymptotic bound

Tomamichel, Hayashi '13
Li '14

$$-\frac{1}{n}\log\beta_n^*(\varepsilon) = S(\rho|\sigma) + \frac{1}{\sqrt{n}}\sqrt{V(\rho|\sigma)}\,\Phi^{-1}(\varepsilon) + O\left(\frac{\log n}{n}\right)$$

$$\Phi(x) \equiv \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} dt\, e^{-t^2/2}$$

- Leading term: quantum Stein lemma
- Gives operational meaning to S and V
- Reminiscent of law of large numbers

Qubit example:

$$\rho = \frac{1}{2}(1 + \vec{a} \cdot \vec{\sigma})$$

$$\sigma = \frac{1}{2}(1 + \vec{b} \cdot \vec{\sigma})$$

$$A = c_0 + \vec{c} \cdot \vec{\sigma}$$

POVM: $A \geq 0 \wedge 1 - A \geq 0 \Rightarrow |\vec{c}| \leq \min(c_0, 1 - c_0)$

Errors: $\alpha = 1 - c_0 - \vec{a} \cdot \vec{c}$

$$\beta = c_0 + \vec{b}.\vec{c}$$

Perturbative setting:

$$\begin{aligned} \rho &= \sigma + \lambda\rho^{(1)} + \frac{\lambda^2}{2}\rho^{(2)} + \mathcal{O}(\lambda^3) \\ S(\rho|\sigma) &= \frac{\lambda^2}{2}S^{(2)}(\rho|\sigma) + \mathcal{O}(\lambda^3) \\ V(\rho|\sigma) &= \frac{\lambda^2}{2}V^{(2)}(\rho|\sigma) + \mathcal{O}(\lambda^3) \end{aligned}$$

Quantum Fisher information:

$$F = \mathrm{Tr}(\sigma L^2), \qquad \rho^{(1)} = \frac{1}{2}(\sigma L + L\sigma)$$

Quantum Cramér-Rao bound:

Braunstein, Caves'94

$$\Delta\lambda^2 \geq \frac{1}{NF(\lambda)}$$

Quantum parameter estimation

Error:

$$-\log \beta_n^*(\varepsilon) = n\frac{\lambda^2}{2} S^{(2)}(\rho|\sigma) + \frac{\lambda\sqrt{n}}{\sqrt{2}}\sqrt{V^{(2)}(\rho|\sigma)}\,\Phi^{-1}(\varepsilon) + O\left(\log n\right)$$

One can show that:

$$2F \leq 2S^{(2)}(\rho|\sigma) \leq V^{(2)}(\rho|\sigma)$$

With equality if and only if $[\sigma, \rho^{(1)}] = 0$

This illustrates that parameter estimates (quantum metrology) and hypothesis testing are closely related.

Measurement strategies

Introduce normalized eigenstates (recall energies are eigenvalues of modular Hamiltonians)

$$
\begin{aligned}
|\mathbf{E}\rangle &= |E_1\rangle \otimes |E_2\rangle \otimes \cdots \otimes |E_n\rangle & \sigma^{\otimes n} \\
|\tilde{\mathbf{E}}\rangle &= |\tilde{E}_1\rangle \otimes |\tilde{E}_2\rangle \otimes \cdots \otimes |\tilde{E}_n\rangle & \rho^{\otimes n}
\end{aligned}
$$

$$
|\mathbf{E}| = \frac{1}{n}\sum_{i=1}^{n} E_i, \qquad |\tilde{\mathbf{E}}| = \frac{1}{n}\sum_{i=1}^{n} \tilde{E}_i
$$

Likelihood ratio test, best strategy in classical case, POVM is projector onto acceptance subspace

$$
\mathcal{H}_C = \operatorname*{span}_{\mathbf{E}} \left\{ |\mathbf{E}\rangle \mid \frac{\log\langle \mathbf{E}|\rho^{\otimes n}|\mathbf{E}\rangle}{\log\langle \mathbf{E}|\sigma^{\otimes n}|\mathbf{E}\rangle} \geq n\mathcal{E} \right\}
$$

Optimal quantum measurement

Li '14

POVM is projection onto the acceptance subspace

$$\mathcal{H}_Q = \operatorname{span}_{\tilde{\mathbf{E}}} \left\{ \sum_{\mathbf{E}:|\mathbf{E}|-|\tilde{\mathbf{E}}|\geq\mathcal{E}} \langle \mathbf{E}|\tilde{\mathbf{E}}\rangle |\mathbf{E}\rangle \right\}$$

Both are hard to implement in practice and involve correlations between then different copies of the density matrices – independent measurements perform poorly.

For applications to spin chains and 2d CFTs, see our paper.

Summary/comments:

- Capacity of entanglement is an interesting probe, sheds light on quasiparticle picture for example, but does not yet shed a lot of light on gravitational fluctuations.
- Found an operational interpretation of relative entropy variance but to connect to gravitational fluctuations need to understand its holographic dual (work in progress)
- Measuring states is clearly of great interest for bulk reconstruction: decoding Hawking radiation, entanglement wedge reconstruction – would be nice to explore in more detail (role for n copies vs replicas?)
- To have n copies of the state may be difficult, could e.g. consider a sequence of evenly space subsystems in a translationally invariant theory.