Logarithmic terms in entanglement entropy: black holes, anomalies and boundaries

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Outline of the talk

- Introduction
- Entanglement entropy of black holes
- Puzzle of non-minimal coupling
- Why log terms might be interesting in case of black holes?
- Log terms in EE of a Conformal Field Theory (CFT)
- What is new when there are boundaries?
- Conclusions

A pure (vacuum) state $|\psi>=\sum_{i,a}\psi_{ia}|A>_i\,|B>_a$ and density matrix $\rho_0(A,B)=|\psi><\psi|$

$|A>$ states are inside surface Σ and $|B>$ are outside of Σ

Density matrix $\rho_B=\mathrm{Tr}\,_A\rho_0(A,B)$ and entropy $S_B=-\mathrm{Tr}\,\rho_B\ln\rho_B$

Since $\mathrm{Tr}\,\rho_A^k=\mathrm{Tr}\,\rho_B^k$ entropy $S_A=S_B$ depends on geometry of separation surface Σ and space-time geometry near Σ

That is why in earlier years it was called *geometric entropy*

Bombelli, Koul, Lee and Sorkin '86; Srednicki '93; Frolov and Novikov '93

In Quantum Field Theory entanglement entropy is UV divergent (function of UV cut-off ϵ)

to leading order EE is proportional to area of Σ

$$S \sim \frac{A(\Sigma)}{\epsilon^{d-2}} \quad if \quad d > 2 \quad \text{and} \quad S \sim \frac{c}{6}\ln(1/\epsilon) \quad if \quad d = 2$$

due to short-distance correlations across Σ

Bombelli, Koul, Lee and Sorkin '86; Srednicki '93; Holzhey, Larsen and Wilzcek '94

In 2d CFT c is central charge $< T >= \frac{c}{48\pi} R$

Introduction: general structure of EE

More generally, in d-dimensional curved space-time (with no boundary) EE is a Laurent series

$$S = \frac{s_{d-2}}{\epsilon^{d-2}} + \frac{s_{d-4}}{\epsilon^{d-4}} + \cdots + \frac{s_{d-2n}}{\epsilon^{d-2n}} + \cdots + s_0 \ln \epsilon + s(g)$$

$$s_{d-2-2n} = \sum_{(l+p)=n} \int_\Sigma \mathcal{R}^l k^{2p}$$

$\mathcal{R}$ is Riemann curvature and k is extrinsic curvature of Σ

Since there are 2 normal vectors to Σ only even powers of k may appear

Logarithmic term s_0 is non-zero if d is even

If space-time has boundary ∂M and if Σ intersects ∂M then the story is different: Log term may appear in any dimension d (odd or even)

In Quantum Field Theory and in presence of rotational symmetry (in sub-space orthogonal to Σ)

$$\mathrm{Tr}\,\rho^n = Z[C_n]$$

is partition function on conical space with angle deficit $2\pi(1-n)$ at surface Σ so that EE is computed by differentiating w.r.t. n of effective action $W(n) = -\ln Z(n)$ on conical space

$$S = (n\partial_n - 1)W(n)|_{n=1}$$

Heat kernel method (field operator $\mathcal{D} = -\nabla^2 + \xi R$)

$$W = -\frac{1}{2}\int_{\epsilon^2}^{\infty}\frac{ds}{s}\mathrm{Tr}\,K(s)\,, \quad \mathrm{Tr}\,K_{\mathcal{M}_n} = \frac{1}{(4\pi s)^{d/2}}\sum_{k=0}(a_k^{reg} + a_k^{\Sigma})s^k$$

$$a_1^{\Sigma} = \frac{\pi}{3}\frac{1-\alpha^2}{\alpha}\int_{\Sigma} 1$$

$$a_2^{\Sigma} = \frac{\pi}{3}\frac{1-\alpha^2}{\alpha}\int_{\Sigma}(\frac{1}{6} - \xi)R - \frac{\pi}{180}\frac{1-\alpha^4}{\alpha^3}\int_{\Sigma}(R_{aa} - 2R_{abab})$$

McKean and Singer '67; Cheeger '83; Dowker '77; Fursaev '94

If no rotational symmetry (extrinsic curvature k of Σ is non-zero) one considers *squashed cones*

$$\int_{\mathcal{M}_n} R = n\int_{\mathcal{M}} R + 4\pi(1-n)\int_\Sigma 1$$

D.D. Sokolov and A. Starobinsky '77

$$\int_{\mathcal{M}_n} R^2 = n\int_{\mathcal{M}} R^2 + 8\pi(1-n)\int_\Sigma R$$

$$\int_{\mathcal{M}_n} R^2_{\mu\nu} = n\int_{\mathcal{M}} R^2_{\mu\nu} + 4\pi(1-n)\int_\Sigma (R_{aa} - \frac{1}{2}k^2)$$

$$\int_{\mathcal{M}_n} R^2_{\alpha\beta\mu\nu} = n\int_{\mathcal{M}} R^2_{\alpha\beta\mu\nu} + 8\pi(1-n)\int_\Sigma (R_{abab} - \mathrm{Tr}\, k^2)$$

$$R_{ab} = R_{\mu\nu} n^\mu_a n^\nu_a \quad \textit{and} \quad R_{abab} = R_{\alpha\beta\mu\nu} n^\alpha_a n^\beta_b n^\mu_a n^\nu_b$$

Fursaev and SS '94; Fursaev, Patrushev and SS '13

Topological Euler number

$$\chi_4[\mathcal{M}_n] = n\chi_4[\mathcal{M}] + (1-n)\chi_2[\Sigma]$$

Conformal invariant

$$\int_{\mathcal{M}_n} W^2 = n\int_{\mathcal{M}} W^2 + 8\pi(1-n)\int_\Sigma [W_{abab} - \mathrm{Tr}\,\hat{k}^2]$$

$\hat{k}^a_{\mu\nu} = k^a_{\mu\nu} - \frac{1}{2}\gamma_{\mu\nu}\mathrm{Tr}\,k^a$, $a = 1, 2$ is conformal invariant constructed from extrinsic curvature.

Fursaev and SS '94; Fursaev, Patrushev and SS '13

Historically the study of EE was motivated by attempts to find a stat. mechanical explanation of Bekenstein-Hawking entropy

If Σ is black hole horizon then its extrinsic curvature vanishes $k^a = 0\,,\ a = 1, 2$

rotational symmetry is generated by Killing vector

$$S_{d=4} = \frac{A(\Sigma)}{48\pi\epsilon^2} - \frac{1}{144\pi}\int_\Sigma [R(1-6\xi) - \frac{1}{5}(R_{aa} - 2R_{abab})]\ln\epsilon$$

SS '94

EE of the Schwarzschild black hole

$$S_{Sch} = \frac{A(\Sigma)}{48\pi\epsilon^2} + \frac{1}{45}\ln\frac{r_+}{\epsilon}$$

SS '94

This is entire entanglement entropy including UV finite part!

Puzzle of non-minimal coupling

If Riemann curvature appears in field operator (as in $\mathcal{D} = -\nabla^2 + \xi R$) should we take into account its distributional part when consider on conical space $\mathcal{M}_n$?

If we do then (for scalar field) one finds for heat kernel

$$a_k^\Sigma \to a_k^\Sigma - 4\pi\xi(1-n)\int_\Sigma \mathbf{a}_{k-1}^{reg}$$

and for entropy (SS '95)

$$S_{con} = \frac{A(\Sigma)}{48\pi\epsilon^2}(1-6\xi) - \frac{1}{144\pi}\int_\Sigma [R(1-6\xi)^2 - \frac{1}{5}(R_{aa} - 2R_{abab})] \ln\epsilon$$

In Log term no changes if $\xi = 1/6$ (conformal case) since $\mathbf{a}_1^{reg} = 0$ in this case

s_0 is invariant under conformal rescaling preserving horizon

Area term is not positive definite in general. That means this entropy does not correspond to a well defined density matrix.

Similar story for gauge fields (contact terms of D. Kabat '95)

$$S_{con} = \frac{A[\Sigma]}{8\pi\epsilon^2}\left(\frac{d-2}{6} - 1\right)$$

is negative in dimensions $d < 8$

Log modification of BH entropy (Fursaev '94; SS '97)

$$S(M) = 4\pi \frac{M^2}{M_{PL}^2} + \sigma \ln M$$

where σ depends on multiplet of fields

$$\sigma = \frac{1}{45}(N_0 + \frac{7}{4}N_{1/2} - 13N_1 - \frac{233}{4}N_{3/2} + 212N_2 + 91N_A)$$

In Standard Model with graviton $\sigma = 16/5$ (without graviton $\sigma = -68/45$)

It produces modification in Hawking temperature

$$1/T_H = 8\pi \frac{M}{M_{PL}^2} + \frac{\sigma}{M}$$

so that $T_H \sim M$ for small black holes

Evaporation rate

$$\frac{dM}{dt} = -T_H^4 M^2$$

If $\sigma > 0$ then black hole evaporation time is infinite (possible consequences for primordial black holes?)

Trace anomaly in $d = 4$

$$< T^\mu_\mu >= -\frac{A}{5760\pi^2} E_4 + \frac{B}{1920\pi^2} W^2$$

$E_4 = R^2_{\mu\nu\alpha\beta} - 4R^2_{\mu\nu} + R^2$ is Euler density

$A_0 = 1\,,\ A_{1/2} = 11\,,\ A_1 = 62 \qquad B_0 = 1\,,\ B_{1/2} = 6\,,\ B_1 = 12$

Proposal for Log term in EE (based on conformal invariance and holography)

$$s_0^{CFT} = \frac{A}{180}\chi[\Sigma] - \frac{B}{240\pi}\int_\Sigma [W_{abab} - \mathrm{Tr}\,\hat{k}^2]$$

SS '08

$\chi[\Sigma]$ is Euler number and $\hat{k}^a$, $a = 1, 2$ is traceless part of extrinsic curvature

Applied for black holes this formula gives:

For extremal black holes (with near horizon geometry $H_2 \times S_2$)

$$s_0 = \frac{A}{90}$$

For the Schwarzschild black holes

$$s_0 = \frac{A - 3B}{90}$$

Note: for $\mathcal{N} = 4$ SYM in our normalization one has that $A = 3B$.

also related works of A. Sen and collaborators '11-'13 on supergravity vs microscopic entropy

Two test geometries in Minkowski spacetime ($W_{abab} = 0$):

$\Sigma = S_2 : \chi = 2, \hat{k}^a = 0 , a = 1, 2$ $\boxed{s_0 = \frac{A}{90}}$

(this case is conformally equivalent to extremal black hole)

$\Sigma = \mathrm{Cylinder}_2 \text{:} \ \chi = 0, \ \mathrm{Tr}\, \hat{k}^2 = \frac{1}{2R^2}$ $\boxed{s_0 = \frac{B}{240} \frac{L}{R}}$

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conformal scalar fields if Σ *is sphere*
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holographic CFT and its deformations (many papers)

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gauge fields if Σ is sphere

$$s_0 = \frac{62}{90}\ (\textit{predicted}) \quad VS \quad s_0 = \frac{32}{90}\ (\textit{calculated})$$

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gauge fields if Σ *is cylinder*

$$s_0 = \frac{12}{240}\frac{L}{R}\ (predicted) \quad VS \quad s_0 = \frac{7}{240}\frac{L}{R}\ (calculated)$$

Huerta and Pedraza '18

(Note: possibly a mistake (private communication from authors)

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$$W_{wedge} = -\frac{1}{2}\int_\Sigma ((\nabla\phi)^2 + \lambda \mathrm{Tr}\,\hat{k}^2\phi^2)$$

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Any way: The proposal works for strongly coupled $\mathcal{N}=4$ SYM. Why it should not work for weakly coupled super-gauge multiplet (scalars, Dirac fermions and gauge fields)?

Increasing activity since 2015:

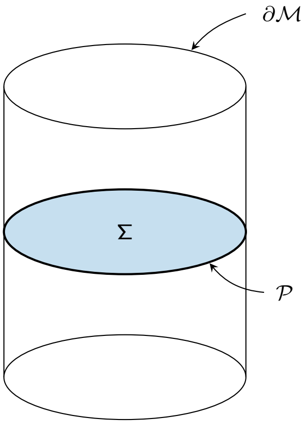
Herzog, Huang, Jensen ('15 and '17); Fursaev ('15), Jensen, O'Bannon ('15); SS ('15), Fursaev, SS ('16); Huang ('16); Berthiere, SS ('16); Astaneh, SS ('17), Astaneh, Fursaev, Berthiere, SS ('17); Herzog, Huang ('17); Chu, Miao, Guo ('17); Rodriguez-Gomez, Russo ('17 and '18); Seminara, Sisti, Tonni ('17 and '18); Berthiere ('18)

A richer structure (yet to be fully uncovered) of Weyl anomaly:

$$\int_{\mathcal{M}_d} \sqrt{g}\, \langle T_{\mu\nu} \rangle\, g^{\mu\nu} = a\chi(\mathcal{M}_d) + b_k \int_{\mathcal{M}_d} \sqrt{\gamma} I_k(W)$$

$$+a'\chi(\partial\mathcal{M}_d) + b'_k \int_{\partial\mathcal{M}_d} \sqrt{\gamma} J_k(W, \hat{K}) + c_n \int_{\partial\mathcal{M}_d} \sqrt{\gamma}\mathcal{K}_n(\hat{K})\,,$$

$\chi[\mathcal{M}_d]$ is Euler number of manifold $\mathcal{M}_d$, $I_k(W)$ are conformal invariants constructed from the Weyl tensor, $\mathcal{K}_n(\hat{K})$ are polynomial of degree $(d-1)$ of the trace-free extrinsic curvature, $K_{\mu\nu} = K_{\mu\nu} - \frac{1}{d-2}\gamma K$ is trace free extrinsic curvature of boundary; $\hat{K}_{\mu\nu} \to e^{\sigma}\hat{K}_{\mu\nu}$ if $g_{\mu\nu} \to e^{\sigma} g_{\mu\nu}$.



$\partial\mathcal{M}$
Σ
$\mathcal{P}$

d = 3 :

$$\int_{\mathcal{M}_3} \langle T \rangle = \frac{c_1}{96}\chi[\partial\mathcal{M}_3] + \frac{c_2}{256\pi}\int_{\partial\mathcal{M}_3} \mathrm{Tr}\,\hat{K}^2$$

Charges (c_1, c_2):

$(-1, 1)$ for scalar filed (Dirichlet b.c.)
$(1, 1)$ for scalar field (conformal Robin b.c)
$(0, 2)$ for Dirac field (mixed b.c.)

A curious observation: for free fields c_2 equals to C_T (that appears in TT 2-point correlation function); is there a general proof that $c_2 = C_T$? or a counter-example?

Log term in entanglement entropy:

$$s_{log} = \frac{c_1}{24}\mathcal{N}$$

$\mathcal{N}$ is number of intersections of Σ and $\partial\mathcal{M}_3$

d = 4 :

$$\int \langle T\rangle = -\frac{a}{180}\chi[\mathcal{M}_4]+\frac{b}{1920\pi^2}\left(\int_{\mathcal{M}_4}\mathrm{Tr}\,W^2 \; - 8\int_{\partial\mathcal{M}_4} W^{\mu\nu\alpha\beta}N_\mu N_\beta \hat{k}_{\nu\alpha}\right)+\frac{c}{280\pi^2}\int_{\partial\mathcal{M}_4}\mathrm{Tr}\,\hat{k}^3$$

$$s_{log} = \frac{a}{720\pi}\left[\int_\Sigma R_\Sigma + 2\int_{\mathcal{P}} k_p\right] - \frac{b}{240\pi}\int_\Sigma [W_{ijij} - \mathrm{Tr}\,\hat{k}_i^2] + d\,F_d + e\,F_e$$

$$\text{where}\quad F_d = -\frac{1}{40\pi}\int_{\mathcal{P}} \hat{k}_{\mu\nu}v^\mu v^\nu \quad F_e = -\frac{1}{\pi}\int_{\mathcal{P}}(N\cdot p_i)(\hat{k}_i)_{\mu\nu}v^\mu v^\nu$$

Theory	a	b	c	d	boundary condition
real scalar	1	1	1	1	Dirichlet
real scalar	1	1	$\frac{7}{9}$	$-\frac{2}{3}$	conformal Robin
Dirac spinor	11	6	5	1	mixed
gauge boson	62	12	8	7	absolute/relative

Complete agreement with holographic computation for $\mathcal{N} = 4$ SYM provided boundary conditions preserve 1/2 SUSY

Astaneh, SS ('17); Astaneh, Berthiere, Fursaev, SS ('17)

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- Many interesting future directions: boundaries, interactions, strings..

Thank you for your attention!