Exotic Vortices and their Dynamics

Nick Manton

DAMTP, University of Cambridge
N.S.Manton@damtp.cam.ac.uk

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1. Abelian Higgs Vortices

- A vortex is a two-dimensional static soliton on a flat or curved surface.The fields are a complex Higgs field ϕ and an electromagnetic potential a.
- The background is a Riemann surface M, with metric

$$ds_0^2 = \Omega_0(z, \bar{z})\, dz d\bar{z}\,.$$

  $z = x_1 + ix_2$ is a (local) complex coordinate, and Ω_0 the conformal factor.
- We consider $N > 0$ vortices. ϕ and a are a section and connection of a $U(1)$ bundle over M, with first Chern number N.

Self-dual vortices

- The vortex equations are

$$
\begin{aligned}
D_{\bar{z}}\phi &\equiv \frac{1}{2}(D_1\phi + iD_2\phi) = 0\,, \\
\frac{1}{\Omega_0} f_{12} &= 1 - \phi\overline{\phi}\,.
\end{aligned}
$$

- $\frac{1}{\Omega_0} f_{12}$ is the physical magnetic field strength on the curved surface.
- A zero of ϕ is a vortex centre.

- We solve $D_{\bar{z}}\phi \equiv \partial_{\bar{z}}\phi - ia_{\bar{z}}\phi = 0$, finding

$$a_{\bar{z}} = -i\partial_{\bar{z}}(\log\phi)\,, \quad a_z = i\partial_z(\log\overline{\phi})\,.$$

- Then

$$f_{12} = -2i\, f_{z\bar{z}} = -2\partial_z\partial_{\bar{z}}\log\phi\overline{\phi} = -\frac{1}{2}\nabla^2\log\phi\overline{\phi}\,.$$

  and the second equation reduces to

$$-\frac{1}{2\Omega_0}\nabla^2\log\phi\overline{\phi} = 1 - \phi\overline{\phi}\,.$$

  Setting $|\phi|^2 = \phi\overline{\phi} = e^{2u}$, we find

$$-\frac{1}{\Omega_0}\nabla^2 u = 1 - e^{2u}\,,$$

  the Taubes vortex equation.
- u has logarithmic singularities at the zeros of ϕ, so there are additional delta functions here.

2. Exotic Vortices

- A more general vortex equation is

$$-\frac{1}{\Omega_0}\nabla^2 u = -C_0 + Ce^{2u}\,.$$

  The constants C_0 and C can be scaled to $-1, 0$ or $+1$.

- For $N > 0$, the RHS needs to be positive for some u.
- The five surviving vortex types are
  (i) Taubes vortices ($C_0 = -1$, $C = -1$);
  (ii) "Bradlow" vortices ($C_0 = -1$, $C = 0$);
  (iii) Ambjørn–Olesen vortices ($C_0 = -1, C = 1$);
  (iv) Jackiw–Pi vortices ($C_0 = 0$, $C = 1$);
  (v) Popov vortices ($C_0 = 1$, $C = 1$).
- Vortex centres are where e^{2u} vanishes. For $C = -1$ the magnetic field is maximal there (Meissner effect); for $C = 1$ it is minimal (antiMeissner effect).

Integrable vortices

- The vortex equations are integrable on backgrounds with Gaussian curvature $K_0 = C_0$. Locally,

$$\Omega_0 = \frac{4}{(1 + C_0|z|^2)^2} \, .$$

  Integrable backgrounds include
  (i) hyperbolic plane for Taubes vortices (Witten),
  (ii) flat plane or torus for Jackiw–Pi vortices (Horvathy and Zhang),
  (iii) sphere for Popov vortices (NSM).

- In integrable cases, the vortex equation reduces to Liouville's equation, and solutions are constructed using a holomorphic function $f(z)$.
- The solution is

$$|\phi|^2 = e^{2u} = \frac{(1 + C_0|z|^2)^2}{(1 + C|f(z)|^2)^2} \left|\frac{df}{dz}\right|^2 .$$

  and one may fix the gauge by choosing

$$\phi = \frac{1 + C_0|z|^2}{1 + C|f(z)|^2} \frac{df}{dz} .$$

- Globally, f is a map from M, with curvature C_0, to a surface with curvature C. $|\phi|^2$ is the ratio of metrics.

- Vortex centres are at ramification points of f, where $\frac{df}{dz} = 0$.
- For Popov vortices on a sphere, f is a rational function of degree n. $\frac{df}{dz}$ then has $N = 2n - 2$ zeros, so N is even.

3. Vortices as Conical Geometry

- Consider the original metric $ds_0^2 = \Omega_0 \, dzd\bar{z}$ and the conformally rescaled Baptista metric

$$ds^2 = \Omega \, dzd\bar{z} = e^{2u} \, \Omega_0 \, dzd\bar{z} \, .$$

The Baptista metric defines an intrinsic geometry of the vortex.

- ds_0^2 and ds^2 are different Riemannian geometries on M. The complex structure and topology are unchanged. But ds^2 has conical singularities. A basic vortex has cone angle 4π (conical excess = 2π), because locally

$$ds^2 = r^2 \, (dr^2 + r^2 d\theta^2) = d\rho^2 + \rho^2 d\chi^2$$

with $\rho = \frac{1}{2} r^2$ and $\chi = 2\theta$.

Baptista curvature relation

- The Gaussian curvatures of ds_0^2 and ds^2 are

$$K_0 = -\frac{1}{2\Omega_0}\nabla^2 \log \Omega_0 \, ,$$

and

$$K = -\frac{1}{2\Omega}\nabla^2 \log \Omega = -\frac{1}{2e^{2u}\Omega_0}\nabla^2 (2u + \log \Omega_0) \, .$$

Therefore (Kazdan and Warner, Troyanov)

$$-\frac{1}{\Omega_0}\nabla^2 u = -K_0 + K e^{2u} \, .$$

a purely geometric identity.

- In addition there is the vortex equation, so

$$-C_0 + Ce^{2u} = -\frac{1}{\Omega_0}\nabla^2 u = -K_0 + Ke^{2u} \,.$$

- Baptista wrote this as a kind of curvature conservation law between the metrics with and without vortices,

$$(K - C)\Omega = (K_0 - C_0)\Omega_0 \,.$$

  (Not algebraic because of formulae for curvatures.)

- $K_0 = C_0$ is the integrable case. Then $K = C$, and the background and Baptista metrics both have constant curvature. The Baptista metric is a 2-d Einstein metric with cosmological constant. The singularities of cone angle 4π imply that there are particles with negative mass present.

Gauss–Bonnet and vortex number

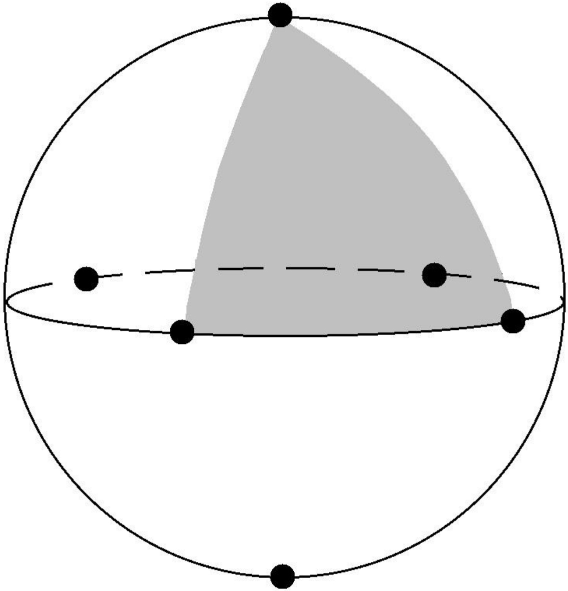
- The Gauss–Bonnet formula taking account of conical singularities constrains the vortex number N.
- Integrating Baptista's conservation law gives

$$CA = C_0 A_0 + 2\pi N$$

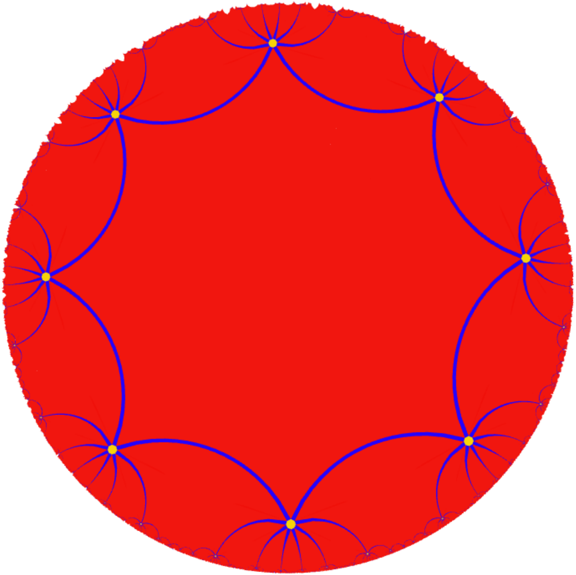
  and the requirement $A > 0$ places bounds on N (Bradlow).
- E.g. for Taubes vortices, $2\pi N < A_0$. More vortices can be squeezed on to M, but not satisfying the Taubes equation.
- For Ambjørn–Olesen vortices, $2\pi N > A_0$.

4. Vortices on the Bolza Surface

- Taubes vortices are integrable on surfaces with $K_0 = -1$. The Baptista metric also has curvature $K = -1$, with conical singularities.
- If M has genus $\mathfrak{g}$, then $A_0 = 2\pi(2\mathfrak{g} - 2)$, so $N < 2\mathfrak{g} - 2$.
- R. Maldonado and NSM constructed an explicit $N = 1$ Taubes vortex on the highly symmetric $\mathfrak{g} = 2$ Bolza surface.
- The Bolza surface is metrically a regular hyperbolic octagon, with opposite sides identified. The vertex angle is $\frac{\pi}{4}$, so the glued, compact surface is smooth.

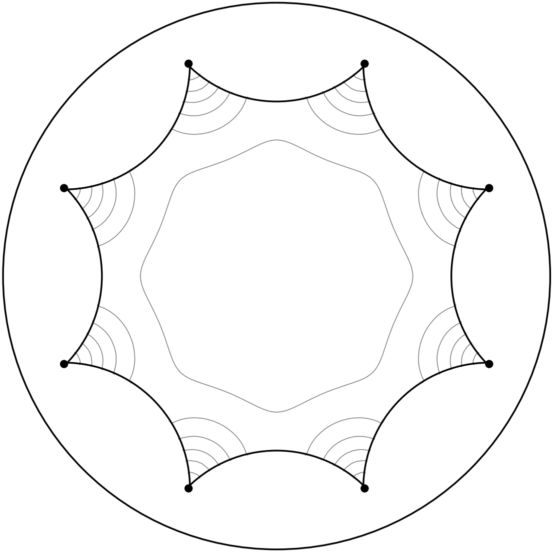


Bolza surface double covers the Riemann sphere

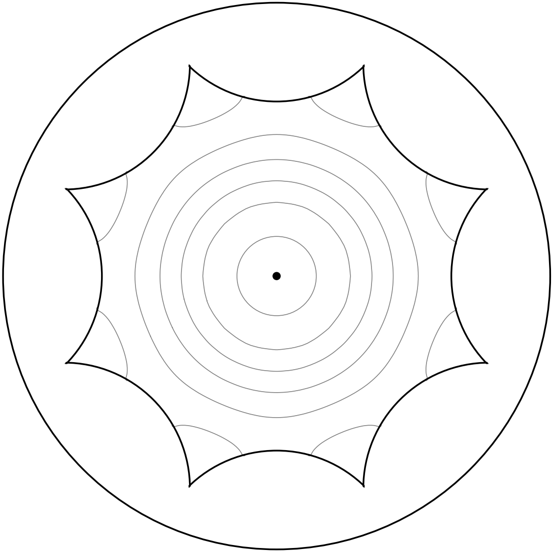


{8,8} tessellation of $\mathbb{H}^2$ by Bolza octagons

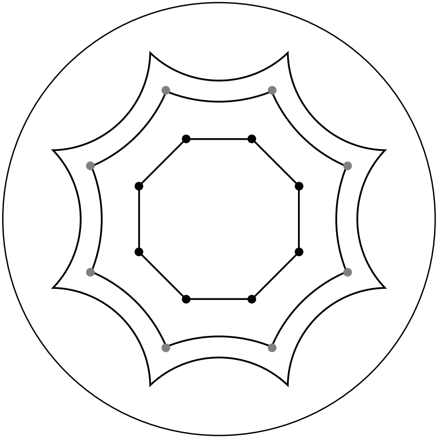
- For the $N = 1$ Taubes vortex at a vertex or centre of the Bolza octagon, $|\phi|^2 = e^{2u}$ depends on Schwarz triangle functions (hypergeometric functions).
- The Baptista geometry has half the area of the Bolza surface, and one cone angle 4π.
- There is a (coincident) $N = 2$ Bradlow vortex on the Bolza surface. The Baptista metric is now flat, with one cone angle 6π.
- There is an $N = 6$ Ambjørn-Olesen vortex on the Bolza surface. The vortices are at the branch points of the double covering of the sphere, and the Baptista metric is the double sphere metric.



Contours of $|\phi|^2 = e^{2u}$ for Bolza vortex at vertex.



Contours of $|\phi|^2 = e^{2u}$ for Bolza vortex at centre.



Bolza octagon (outer) superimposed on the Poincaré disc; Baptista octagon (middle) of an $N = 1$ Taubes vortex; flat Baptista octagon (inner) of an $N = 2$ Bradlow vortex. In all cases, opposite edges are identified.

5. Energy and Dynamics

- The static energy function for all the vortex types we have considered is

$$E = \int_M \left\{ \frac{1}{\Omega_0^2} f_{12}^2 - \frac{2C}{\Omega_0} \left(\overline{D_1\phi} D_1\phi + \overline{D_2\phi} D_2\phi \right) + \left(-C_0 + C|\phi|^2 \right)^2 \right\} .$$

  E is positive definite for $C \leq 0$, but not otherwise. Not all vortex types are stable.
- The Bogomolny argument (completing the squares in E) shows that vortices are stationary points of E, but not necessarily minima.

- The static energy extends to a Lagrangian for fields on $\mathbb{R} \times M$ (metric $dt^2 - \Omega_0 \, dz d\bar{z}$)

$$L = \int_M \left\{ -\frac{1}{2} f_{\mu\nu} f^{\mu\nu} - 2C \overline{D_\mu \phi} D^\mu \phi - \left(-C_0 + C|\phi|^2 \right)^2 \right\} .$$

- The kinetic energy is exotic if $C \geq 0$, as the time derivative of ϕ contributes with a minus sign. But the electric field contribution always has standard form.
- This Lagrangian is a dimensional reduction of pure Yang–Mills theory in $4 + 1$ dimensions (F. Contatto and M. Dunajski). The gauge group may be non-compact (e.g. SU(1,1) or E_2) and this leads to minus signs.

- I have started (with E. Walton) to study the moduli space dynamics of these five vortex types. The kinetic energy of moving vortices simplifies to localised contributions from the vortex centres (Strachan/Samols localisation).
- For exotic vortices the kinetic energy may be zero or negative. Geodesic motion could be along null curves.
- E.g. A Popov vortex at $z = 0$ is described by the rational function $f(z; t) = c(t)z^2$. The fields vary as c varies, but the kinetic energy is zero. This follows from the localisation formula, and can be checked by direct integration.

6. Summary

- There are five variants of Abelian Higgs vortices. They are nicely understood using the Baptista metric $\Omega\, dzd\bar{z} = e^{2u}\,\Omega_0\, dzd\bar{z}$, where a vortex becomes a conical singularity with cone angle 4π.
- On constant curvature backgrounds, vortex equations reduce to Liouville's equation, and solutions are found using holomorphic maps $f(z)$.
- The metric on the vortex moduli space simplifies using Strachan/Samols localisation. Question: For integrable vortices, is this metric a Weil–Petersson metric for moving conical singularities on a surface?
- It would also be interesting to study exotic vortex motion numerically.

Selected references

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