Quantising Gravitational Instantons

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QUANTIZING GRAVITATIONAL INSTANTONS *

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Gravitational instantons are complete non-singular positive definite metrics satisfying the Einstein equations with or without cosmological constant. We treat quantum fluctuations about such solutions and introduce a useful decomposition of the space of gravitational degrees of freedom in terms of which the one-loop corrections take an especially simple form. We treat in detail the question of zero modes, and the fluctuations about de Sitter space.

This talk:

Somewhat unusual description of gravitational instantons and their quantisation

Gravitational instantons here:

Riemannian signature metrics

that are Einstein (with non-zero Λ)

and have half of Weyl curvature zero

$$\text{Riemann} = \begin{pmatrix} \frac{\Lambda}{3}\mathbb{I} + W^+ & \text{Ricci}|_{tf} \\ \text{Ricci}|_{tf} & \frac{\Lambda}{3}\mathbb{I} + W^- \end{pmatrix}$$

Gravitational instantons: $\text{Ricci}|_{tf} = 0, \qquad W^+ = 0$

Message I:

Formalism that describes gravitational instantons in very much the same way as YM instantons are usually described

SDYM and SDGR are both integrable theories related to full YM and full GR

This point of view may be useful to understand both theories better

W.A. Bardeen, "Selfdual Yang-Mills theory, integrability and multiparton amplitudes," Prog. Theor. Phys. Suppl. 123, 1 (1996)

Gravity is more like YM theory in this formalism

Message II:

Both SDYM and SDGR can be quantised

= their quantum effective action computed

Both SDYM and SDGR are one-loop exact

Both have quantum divergences, but for both these can be removed by field redefinitions

Both SDYM and SDGR are quantum finite

But SDGR is power-counting non-renormalisable

Hence an example of a power-counting non-renormalisable theory of spin 2 that is actually quantum finite (and non-trivial)

Self-Dual Yang-Mills

$$S^{SDYM}[A, B^+] = \int \mathrm{Tr}(B^+ \wedge F)$$

Chalmers-Siegel ' 96

$F = dA + A \wedge A$ curvature

B^+ Lie algebra valued self-dual 2-form $\ {}^*B^+ = B^+$

$$B^+_{\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu}{}^{\rho\sigma} B^+_{\rho\sigma}$$

Field Equations

$F^+ = 0$ connection is an instanton

A affects B, B does not affect A

$d_A B^+ = 0$ first order linear PDE on an instanton background

Significance of instantons

any 2-form can be decomposed into SD, ASD parts

$$F = F^+ + F^-$$

half-flat connections

$$F^+ = 0 \quad \Leftrightarrow \quad F = F^- \quad \Leftrightarrow \quad {}^*F = -F$$

when $F^+ = 0$ curvature is pure ASD

such connections are automatically solutions of YM eqs

$$d_A{}^*F = -d_A F = 0$$

instanton condition is the first-order PDE that implies the YM second-order PDE

Instantons are absolute minima of the YM action

$$S^{YM}[A] = \int |F|^2 = \int (|F^+|^2 + |F^-|^2)$$

Over a compact 4-manifold, the bundle is characterised in particular by its first Pontryagin number

$$p_1 = \int \mathrm{Tr}(F \wedge F) = \int (|F^+|^2 - |F^-|^2)$$

depends on the bundle, but not on the connection

$$S^{YM}[A] = -p_1 + 2\int |F^+|^2 \geq -p_1$$

action always greater than Pontryagin

Linearisation of the SDYM action

$$\delta B^+ \equiv b, \qquad \delta A \equiv a$$

background

$$F^+ = 0$$
$$B^+ = 0$$

$$\mathcal{L}^{(2)} = \mathrm{Tr}(b\, d_A a)$$

$$\mathcal{L}^{(3)} = \mathrm{Tr}(baa)$$

cubic non-derivative interaction

Can also linearise around non-zero B, but nothing interesting this way

Spinors

It is very convenient to rewrite in spinor notations

$$\mu \to MM'$$

connection perturbation $a_\mu \to a_{MM'}$

self-dual 2-form perturbation $b_{\mu\nu} \to b_{MN}$

$$b_{MN} = b_{(MN)}$$

$$\mathcal{L}^{(2)} = \mathrm{Tr}(b^{MN} d_M{}^{M'} a_{NM'})$$

$$d^{MM'} \equiv d_A^{MM'}$$ covariant derivative

the derivative operator here maps

$d : S_+ \times S_- \times \mathfrak{g} \to S_+^2 \times \mathfrak{g}$ and is degenerate - gauge

Gauge-fixing

Let us use the sharp (Landau) gauge

gauge-fixing fermion

$$\Psi = \mathrm{Tr}(\bar{c}\, d^{MM'} a_{MM'}) = \mathrm{Tr}(\bar{c}\, \epsilon^{NM} d_M{}^{M'} a_{NM'})$$

$$s\Psi = \mathrm{Tr}(h_c\, \epsilon^{NM} d_M{}^{M'} a_{NM'}) - \mathrm{Tr}(\bar{c}\, d^{MM'} d_{MM'} c)$$

gauge-fixing term ↗ ghost Lagrangian ↖

can now combine the ghost auxiliary field with b

$$\tilde{b}^{MN} = b^{MN} + h_c \epsilon^{NM} \qquad \tilde{b} \in S_+ \times S_+$$

Gauge-fixed Lagrangian

$$\mathcal{L}^{(2)}_{g.f.} = \mathrm{Tr}(\tilde{b}^{MN} d_M{}^{M'} a_{NM'})$$

now the derivative operator maps

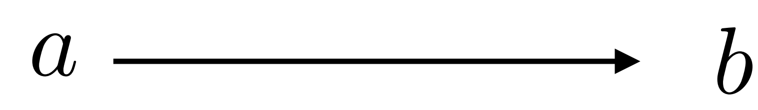
$$d \equiv \not{d} : S_- \times V \to S_+ \times V \qquad\qquad V \equiv S_+ \times \mathfrak{g}$$

standard (chiral) Dirac operator

Perturbation theory

Propagator

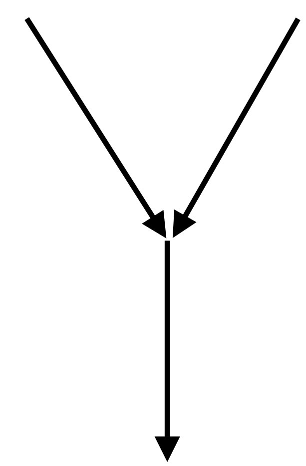
$$\langle b\, a \rangle = \frac{1}{\not{d}}$$



propagator is an oriented line

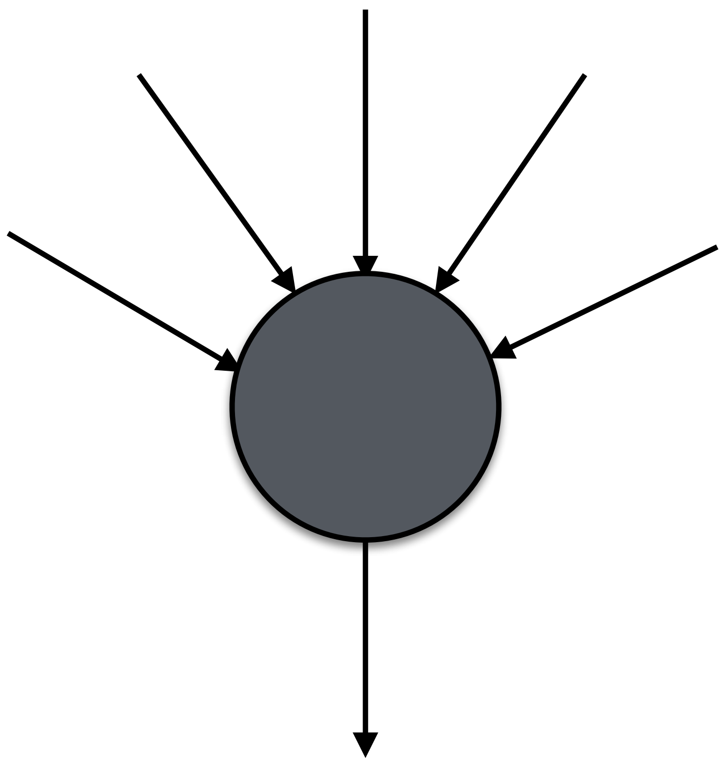
Vertex

$$\langle b\, aa \rangle$$



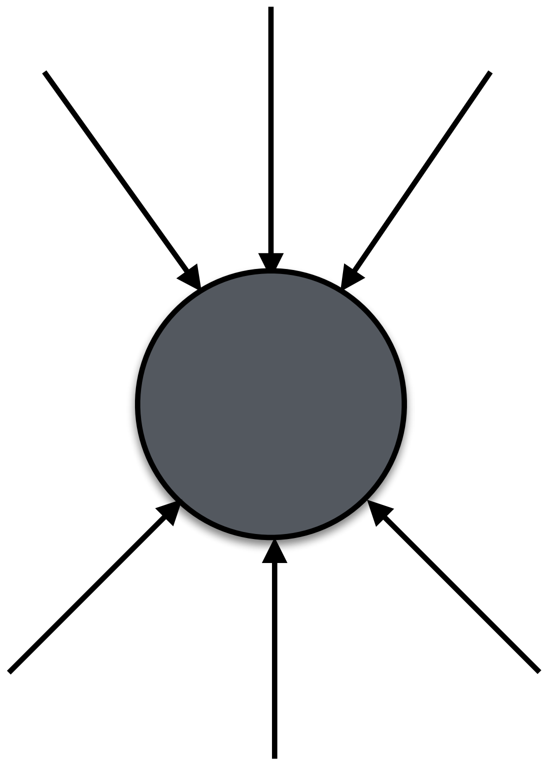
2 a lines in one b line out

Tree - level diagrams



as many external a lines as desired,
only one external b line

One-loop diagrams



no external b lines

No higher loops - one-loop exact!

One-loop effective action

Use background field method

$$\mathcal{L}^{(2)} = \frac{1}{2}\begin{pmatrix} b & a \end{pmatrix}\begin{pmatrix} 0 & \not{d} \\ \not{d}^* & 0 \end{pmatrix}\begin{pmatrix} b \\ a \end{pmatrix} \equiv \frac{1}{2}\begin{pmatrix} b & a \end{pmatrix} D \begin{pmatrix} b \\ a \end{pmatrix}$$

Effective action

from ghosts

$$S_{eff} = -\frac{1}{2}\log\det(D) + \log\det(d^2)$$

Convenient to square the operator to get

$$S_{eff} = -\frac{1}{4}\log\det(D^2) + \log\det(d^2)$$

$$D^2 = \begin{pmatrix} \not{d}^*\not{d} & 0 \\ 0 & \not{d}\,\not{d}^* \end{pmatrix}$$

Dirac squared - Laplacian

Divergences

can be computed using the standard heat kernel expansion

However, on instanton background no computation is needed

divergences are proportional to integral of curvature squared

On instantons

$$\int |F|^2 = -p_1 + 2\int |F^+|^2 = -p_1$$

On instantons the only divergence is a multiple of the first Pontryagin number - does not contribute to S-matrix

Theory is quantum finite

One-loop scattering amplitudes

However, the finite part of the one-loop effective action is non-zero

$$\mathcal{A}^{----} \sim \frac{[12][34]}{\langle 12\rangle\langle 34\rangle}$$

A closed form expression for the general n-point one-loop amplitude exists

Bern et al ‘93

see below for meaning of amplitudes in SDYM

Relation to full YM

$$S^{YM}[A, B^+] = \int \mathrm{tr}(B^+ \wedge F - g^2 B^+ \wedge B^+)$$

Integrating out B^+

first Pontryagin

$$S[A] = \frac{1}{4g^2} \int |F^+|^2 = \frac{1}{4g^2} \left(\int |F|^2 + p_1 \right)$$

So, usual YM plus topological term

Get formalism for QCD with just cubic interactions!

In Lorentzian signature the Lagrangian is not manifestly Hermitian because B^+ is complex

Nevertheless the theory is unitary, and completely equivalent to the usual second-order YM

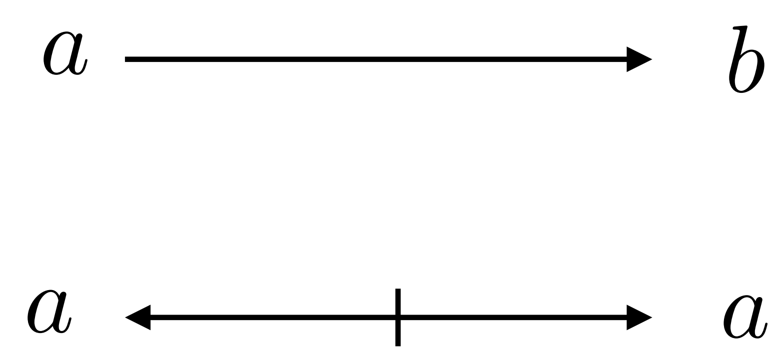
Full YM Feynman rules

Propagators of two types

$$\langle b\, a \rangle = \frac{1}{\not{d}}$$

$$\langle a \;\; a \rangle = \frac{g^2}{\Delta}$$

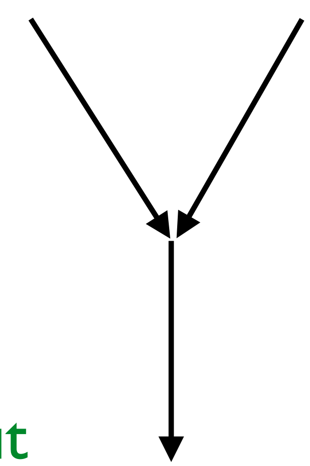
oriented line



orientation reversing line

Vertex

$$\langle b\, aa \rangle$$



2 a lines in single b line out

Many more diagrams, not one-loop exact anymore

Diagrams with the minimum possible number of external b-lines are the same in full YM and SDYM

Some amplitudes at tree level and one-loop are the same in full YM and SDYM

Summary of SD Yang-Mills

One of the fields enters the Lagrangian linearly

$\Rightarrow$ one-loop exact

No divergences - theory is quantum finite

argument involves Pontryagin number

One-loop effective action is non-trivial

Also a powerful cubic formalism for full YM

Bardeen: integrability of SDYM and its close relationship with full YM may be useful to understand the latter

Self-Dual Gravity

arXiv:1610.01457

The only previously known covariant formulation of SDGR was breaking one of self-dual halves of Lorentz

Siegel

Light-cone non-covariant versions of SDGR were also available

I will describe a somewhat unconventional version which uses connections rather than metrics

This description of instantons was discovered in 70’s but never before used in the construction of the quantum theory

The Lagrangian

$$S^{SDGR}[A, \Psi] = \int \Psi^{ij} F^i \wedge F^j$$

$$F^i = dA^i + (1/2)\epsilon^{ijk} A^j \wedge A^k$$

curvature of an SO(3) connection A^i

$\Psi^{ij} = \Psi^{(ij)}$ symmetric tracefree field $\Psi^{ij}\delta_{ij} = 0$

Field equations

this description of instantons not new - 80's - 90's

$$F^i \wedge F^j \sim \delta^{ij}$$ algebraic equation on the curvature

$$d_A \Psi^{ij} = 0$$ first order PDE on the auxiliary field

Claim: describes gravitational instantons with non-zero Λ

How connections can describe metrics

There is a fundamental fact of 4D geometry:

A generic triple of 2-forms in 4D defines a metric

$$(\xi, \eta)_g (\mathrm{vol})_g = \frac{1}{6} \epsilon^{ijk} i_\xi B^i \wedge i_\eta B^j \wedge B^k$$

2-forms become self-dual with respect to this metric

Geometry in 4D can be coded into 2-forms

Can describe GR in this way

Connection description of instantons

Theorem: Gindikin ’82, Capovilla, Dell,Jacobson ‘90

Let A be an SO(3) connection whose curvature satisfies

$$F^i \wedge F^j \sim \delta^{ij}$$

Then the metric constructed from F’s is ASD Einstein with non-zero scalar curvature

A gets identified with the self-dual part of the Levi-Civita connection compatible with this metric

Linearisation

Background $\Psi = 0$ $\delta A = a,$ $M^2 \delta\Psi = \psi$

$$F^i = M^2 \Sigma^i \qquad M^2 = \Lambda/3$$

Σ^i 't Hooft symbols for the background metric "orthonormal" basis of SD 2-forms

$$\Sigma^i_{\mu}{}^{\rho} \Sigma^j_{\nu\rho} = g_{\mu\nu}\delta^{ij} + \epsilon^{ijk}\Sigma^k_{\mu\nu}$$

$$\mathcal{L}^{(2)} = 2\psi^{ij}\Sigma^i d_A a^j$$

negative mass dimension coupling

$$\mathcal{L}^{(3)} = \frac{1}{M^2}\psi^{ij} d_A a^i d_A a^j + \psi^{ij}\Sigma^i \epsilon^{jkl} a^k a^l$$

$$\mathcal{L}^{(4)} = \frac{1}{M^2}\psi^{ij} d_A a^i \epsilon^{jkl} a^k a^l$$

dimensionfull coupling from the background, as in GR

no quintic interaction

Spinors

$$\psi^{ij} \to \psi^{ABCD} \qquad \in S_+^4$$

$$a^i_\mu \to a_{ABMM'} \qquad \in S_+^2 \times S_+ \times S_-$$

$$\mathcal{L}^{(2)} = \psi^{ABCD} d_A{}^{M'} a_{BCDM'}$$

compare SDYM

The operator here maps

$$S_+^2 \times S_+ \times S_- \to S_+^4$$

$$12 \to 5$$

degenerate due to gauge

Gauge

$$a_{ABCM'} \in S_+^2 \times S_+ \times S_- = S_+^3 \times S_- \oplus S_+ \times S_-$$

$$\delta_\xi a_{ABCM'} = \epsilon_{C(A}\xi_{B)M'}$$

pure diffeomorphism gauge

gauge-fixed by setting this part to zero

$$a_{ABCM'} \in S_+^3 \times S_-$$

SO(3) gauge is fixed in the "usual" way

Landau sharp gauge

$$\Psi = \bar{c}^{AB} d^{MM'} a_{ABMM'} \equiv \bar{c}^{AB} \epsilon^{NM} d_M{}^{M'} a_{ABNM'}$$

So, as in YM let us combine

$$\tilde{\psi}^{ABMN} = \psi^{ABMN} + h_c^{AB} \epsilon^{NM} \qquad \in S_+^3 \times S_+$$

Gauge-fixed action

$$\mathcal{L}^{(2)}_{g.f.} = \tilde{\psi}^{ABMN} d_M{}^{M'} a_{ABNM'}$$

the operator here maps

$$\not{d} : S^3_+ \times S_- \to S^3_+ \times S_+$$

the usual (chiral) Dirac operator, all gauge is fixed

Facts about perturbation theory

most of the SDYM facts are still true here:

at tree level, any number of a legs, one ψ leg

one-loop exact

Quantum finiteness

The computation of the effective action reduces to the computation of the determinant of Dirac squared

Divergent parts can be computed by heat kernel

But even without any computation, logarithmic divergences are integrals of curvature squared

Need to analyse all possible invariants that can arise

Euler characteristic

$$\chi = \frac{1}{32\pi^2}\int \left((R_{\mu\nu\rho\sigma})^2 - 4(R_{\mu\nu})^2 + R^2\right)$$

Irreducible parts of the curvature

$$R_{\mu\nu\rho\sigma} = W_{\mu\nu\rho\sigma} + (g_{\mu[\rho}Z_{\sigma]\nu} - g_{\nu[\rho}Z_{\sigma]\mu}) + \frac{R}{6}g_{\mu[\rho}g_{\sigma]\nu}$$

$$Z_{\mu\nu} := R_{\mu\nu} - \frac{R}{4}g_{\mu\nu}$$

then

$$(R_{\mu\nu\rho\sigma})^2 = (W_{\mu\nu\rho\sigma})^2 + 2(Z_{\mu\nu})^2 + \frac{R^2}{6}$$

So

$$\chi = \frac{1}{32\pi^2}\int \left(|W^+|^2 + |W^-|^2 - 2|Z|^2 + \frac{R^2}{6}\right)$$

Signature

$$\tau = \frac{1}{48\pi^2} \int (|W^+|^2 - |W^-|^2)$$

On an instanton background $Z = 0, W^+ = 0$

The two remaining curvature squared invariants

$$\tau = -\frac{1}{48\pi^2} \int (|W^-|^2)$$

$$2\chi + 3\tau = \frac{1}{16\pi^2} \int R^2$$

the only possible divergences are topological

different, higher dimensional interaction, same mechanism as in YM

SD Gravity is quantum finite

Relation to full GR

the Lagrangian for GR in "connection" description

$$S^{GR}[A,\Psi] = -\frac{M_p^2}{2M^2}\int \left((\mathbb{I}+\Psi)^{-1}\right)^{ij} F^i \wedge F^j$$

$$= \frac{M_p^2}{2M^2}\int \left(-\mathbb{I} + \Psi - \Psi^2 + \ldots\right)^{ij} F^i \wedge F^j$$

topological term　　SDGR　　infinite number of higher order terms

like in SDYM, the propagator has an extra part $\langle aa \rangle$

theory is no longer one-loop exact

Summary

An example of power-counting non-renormalisable theory with propagating spin 2 DOF, non-trivial S-matrix but nevertheless quantum finite!

Example of a gravitational theory behaving in exactly the same way as more conventional gauge theory

A novel way of describing spin 2 particles (generalisable to full GR)

SDGR behaves in precise parallel to SDYM

We are not surprised that SDYM is quantum finite

But it is essentially the same mechanism that is responsible for the quantum finiteness of SDGR

Gives one more confirmation of the pattern that gravity is not much different from YM, in spite of its dimension full coupling

Gravity = YM^2 Zvi Bern

How is it possible that the "worse" quantum theory ever - quantum gravity - is so closely related to the "best" quantum theory ever - quantum YM?

Thank you!