EISENHART-DUVAL LIFT

CARROLL SYMMETRY

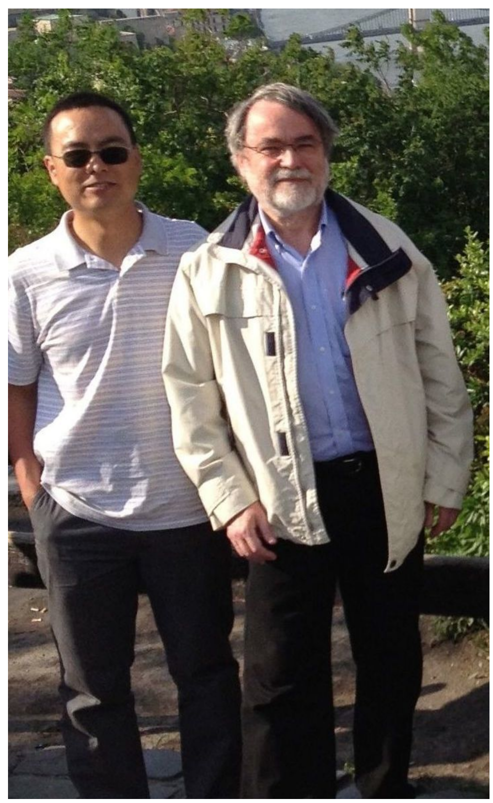
&

GRAVITATIONAL WAVES

Garyfest, March 2017

P. Horvathy, LMPT (Tours)

Azay-le-Rideau, 2014



Budapest 2012

Whereas the usual Wigner-Inönü contraction $c \to \infty$ of the Poincaré group yields the Galilei group, another $c \to 0$ contraction yields the "Carroll group" of Lévy-Leblond. Both boost-invariant theories are conveniently unified within the "Eisenhart-Duval" framework. Plane gravitational waves carry a Carroll symmetry with broken rotations.

Based on:

- C. Duval, G. W. Gibbons, and P. A. Horvathy : *"Celestial Mechanics, Conformal Structures and Gravitational Waves,"* Phys. Rev. **D43**, 3907 (1991)

- C. Duval, G. W. Gibbons, P. A. Horvathy and P. M. Zhang: *"Carroll versus Newton and Galilei: two dual non-Einsteinian concepts of time,"* Class. Quant. Grav. **31** (2014) 085016

- C. Duval, G. W. Gibbons, P. A. Horvathy and P. M. Zhang: *"Carroll symmetry of gravitational plane waves,"* [arXiv:1702.08284 [gr-qc]]

- PM. Zhang, C. Duval, G. W. Gibbons, P. Horvathy: *"Memory effect of gravitational plane waves,"* (work in progress)

Carroll group

Lewis Carroll *Through the Looking Glass and what Alice Found There* (1871).

Carroll group first constructed as novel type of contraction of Poincaré group $\mathrm{E}(d, 1)$:

J. M. Lévy-Leblond, "Une nouvelle limite non-relativiste du group de Poincaré," Ann. Inst. H. Poincaré **3** (1965) 1

V. D. Sen Gupta, "On an Analogue of the Galileo Group," Il Nuovo Cimento **44** (1966) 512

J. Gomis, G Rousseaux, E. Bergshoeff . . .

$$G = -dx^0 dx^0 + \delta_{AB}\, dx^A dx^B. \tag{1}$$

Define time coordinate by

$$t = x^0/c \tag{2}$$

$c \uparrow \infty \rightsquigarrow$ NEWTON-CARTAN STRUCTURE

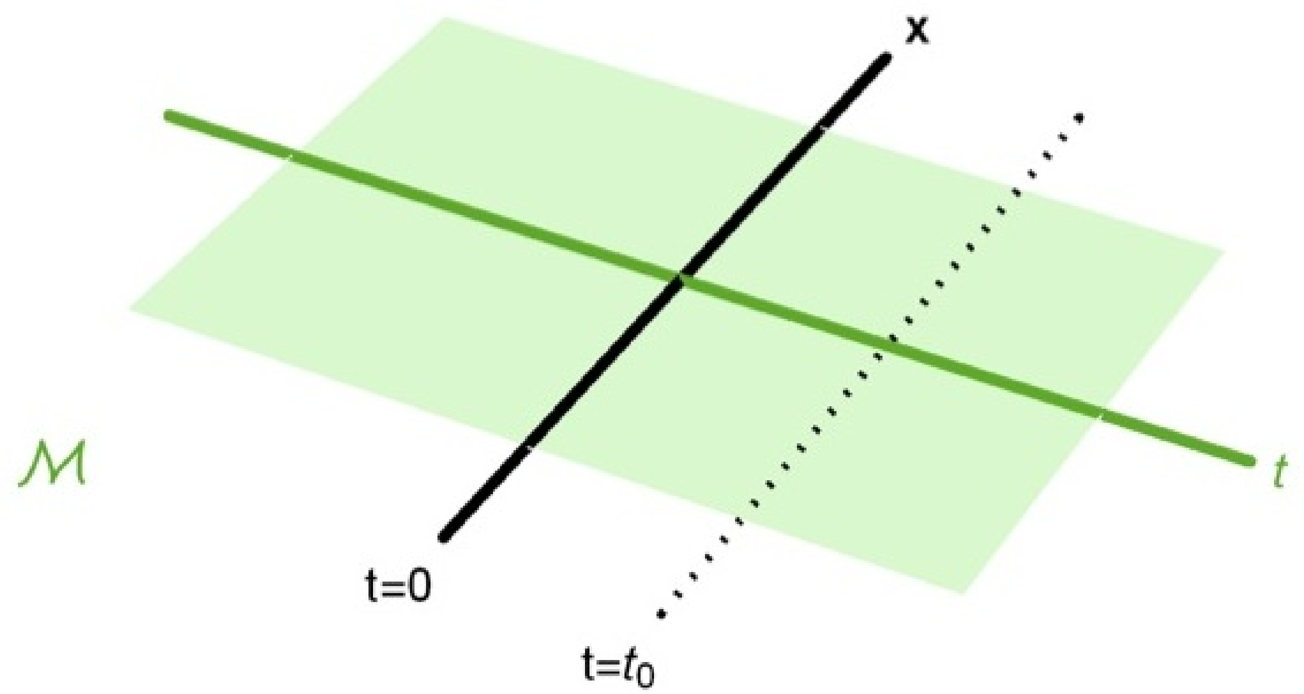


Fig. 1 : Galilean space-time, $\mathcal{M}$, described by $\begin{pmatrix} \boldsymbol{x} \\ t \end{pmatrix}$. Carries symmetric, contravariant non-negative [space-co-] "metric" tensor γ, whose kernel is generated by dt, – i.e., a Newton-Cartan structure. Projects onto absolute time axis.

CARROLL STRUCTURE

Lévy-Leblond  1965 : consider instead novel "time" coordinate, s,

$$s = Cx^0 \tag{3}$$

for some *new constant* C [has again dim of velocity; $[s]$ has dimension of (squared length)/time, *action/mass* .

Minkowski metric (1) written as

$$G = \boxed{-\frac{1}{C^2}\,dsds} + \delta_{AB}\,dx^A dx^B. \tag{4}$$

Carrollian limit $C \uparrow \infty$ yields another degenerate "metric",

$$\boxed{G \to \delta_{AB}\,dx^A dx^B \equiv \bar{G}.} \tag{5}$$

Kernel generated by $\xi = \partial/\partial s$. Manifold with such structure $(\bar{G}, \xi)$:

Carroll space-time, $\mathcal{C} \equiv \mathcal{C}^{d+1}$.

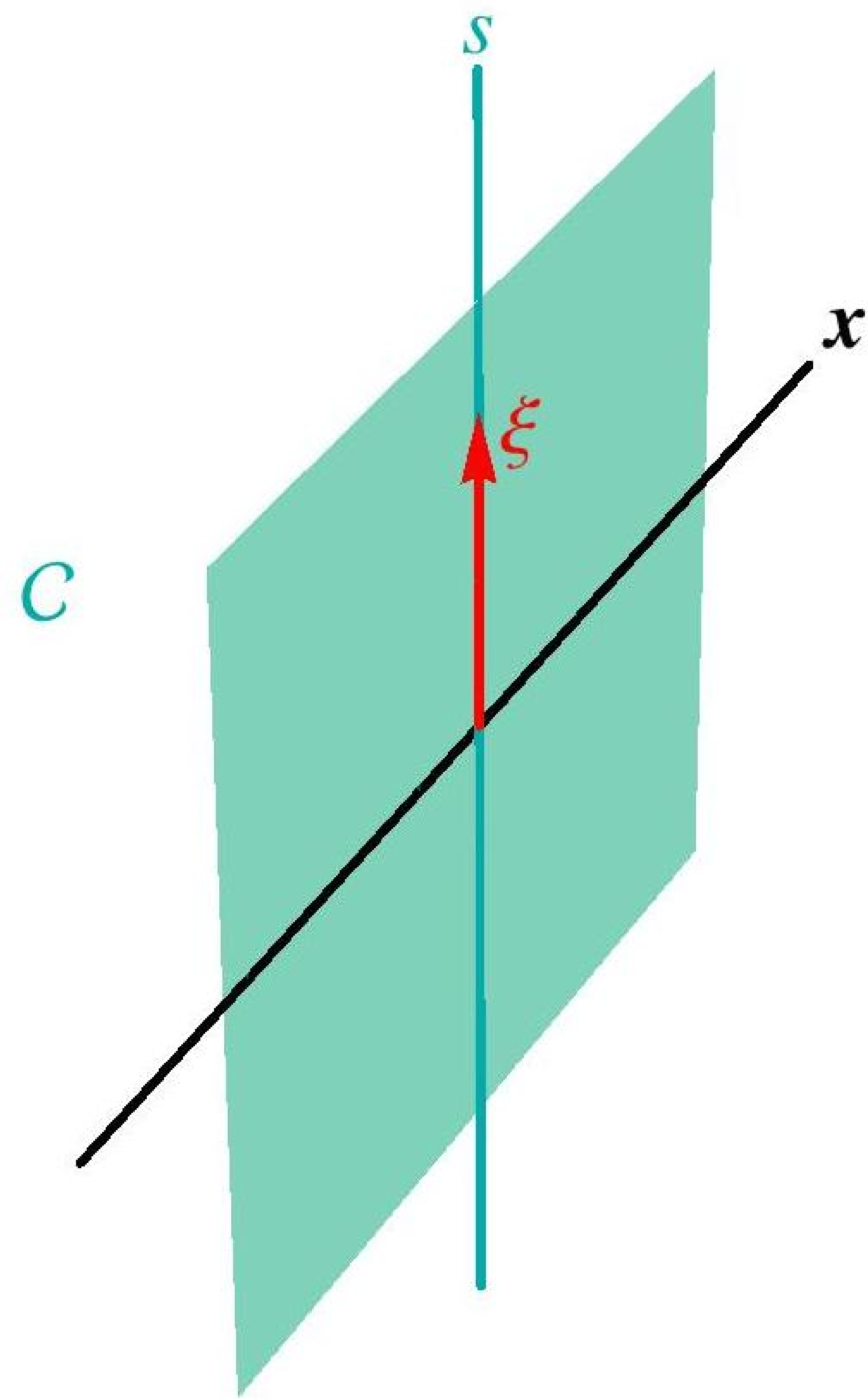


Fig.2 : Carroll space-time $\mathcal{C}$ described by $\begin{pmatrix} \boldsymbol{x} \\ s \end{pmatrix}$ is endowed with vector ξ which generates kernel of (singular) [space-] "metric" $\bar{G}$.

Lévy-Leblond : *Carroll group* Carr$(d+1)$, obtained from orthochronous Poincaré group, $E_+(d,1)$, by contraction $\rightsquigarrow$ end up, in limit $C \uparrow \infty$, with *"Carroll boosts"*

$$\boxed{\begin{cases} \boldsymbol{x}' &= \boldsymbol{x} \\ s' &= s - \boldsymbol{b} \cdot \boldsymbol{x} \end{cases}} \qquad (6)$$

• Carrollian limit of relativistic time-translations: $\boldsymbol{x}' = \boldsymbol{x}$, and $x^{0\prime} = x^0 + a^0 \rightsquigarrow$ Carrollian *"time"-translations*

$$\begin{cases} \boldsymbol{x}' &= \boldsymbol{x}, \\ s' &= s + f \end{cases} \qquad (7)$$

with $f = Ca^0$.

N.B. : In QM wave fct transforms according to:

$$\boxed{\psi'(x,t) = e^{i(-\mathbf{b}\cdot\boldsymbol{x} - \frac{1}{2}\mathbf{b}^2 t)}\psi(\boldsymbol{x} + \mathbf{b}t, t)} \qquad (8)$$

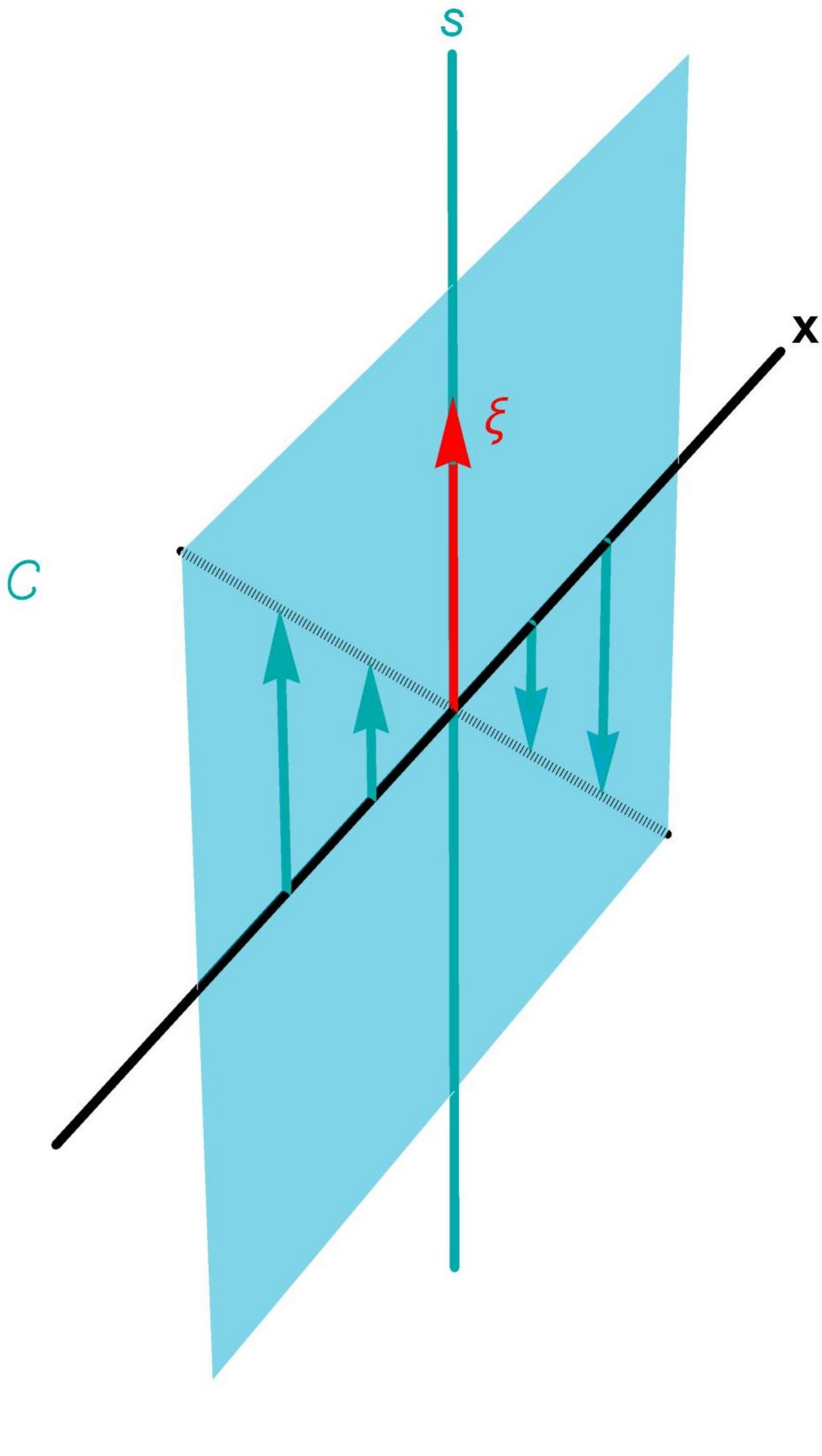


Fig.3a Carroll boosts $\boldsymbol{x}' = \boldsymbol{x}$, $s' = s - \boldsymbol{b} \cdot \boldsymbol{x}$ acting on flat Carroll space-time

Carroll group thus generated by

1. "C-boosts" (6), $s' = s - \boldsymbol{b} \cdot \boldsymbol{x}$

2. orthog. transf $R \in \mathrm{O}(d)$: $\boldsymbol{x}' = R\,\boldsymbol{x}$, $s' = s$,

3. space translations (not affected by contraction), completed with $s' = s$,

4. "C-time"-transl (7), $s' = s + f$, completed with $\boldsymbol{x}' = \boldsymbol{x}$.

Represented by matrices

$$\begin{pmatrix} R & 0 & \mathbf{c} \\ -\mathbf{b}^T R & 1 & f \\ 0 & 0 & 1 \end{pmatrix} \tag{9}$$

where $R \in \mathrm{O}(d)$, $\mathbf{b}, \mathbf{c} \in \mathbb{R}^d$, $f \in \mathbb{R}$. Acts on Carroll space-time affinely by matrix action.

Lie algebra $\mathfrak{carr}(d+1)$ acts on Carroll space-time as

$$X = (\omega^A_B\, x^B + \gamma^A)\frac{\partial}{\partial x^A} + \left(\varphi \;\boxed{-\boldsymbol{\beta}_A\, \boldsymbol{x}^A}\right)\frac{\partial}{\partial s}, \tag{10}$$

where $\omega \in \mathfrak{so}(d)$, $\boldsymbol{\beta}, \boldsymbol{\gamma} \in \mathbb{R}^d$, and $\boldsymbol{\varphi} \in \mathbb{R}$.

Comparison: Galilean time $t = x^0/c$, $\boldsymbol{b} = c\boldsymbol{\beta}$,

$\rightsquigarrow$ in limit $c \uparrow \infty$, ordinary *Galilei boosts*

$$\left\{ \begin{array}{rcl} \boldsymbol{x}' & = & \boldsymbol{x} + \boldsymbol{b}\,t \\ t' & = & t \end{array} \right. \qquad (11)$$

Galilei group obtained by contraction $c \uparrow \infty$.

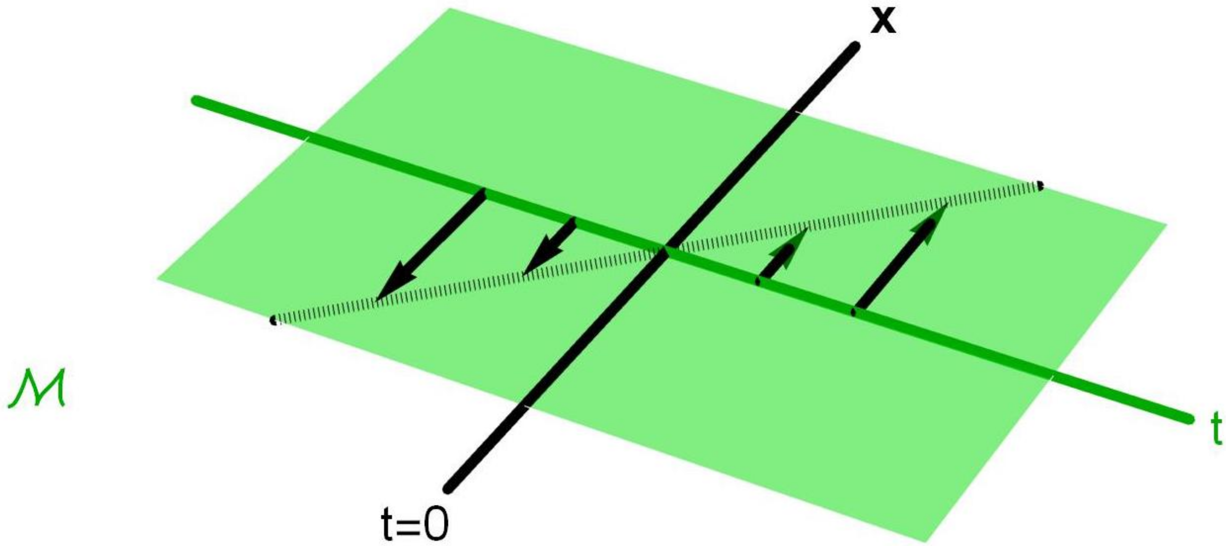


Fig.3b Galilei boost acting on Galilei space-time

Galilei Lie algebra $\mathfrak{gal} \equiv \mathfrak{gal}(d+1)$

$$X = (\omega^A_B x^B + \boxed{\beta^A \mathrm{t}} + \gamma^A)\frac{\partial}{\partial x^A} + \epsilon \frac{\partial}{\partial t} \qquad (12)$$

where $\omega \in \mathfrak{so}(d)$, $\boldsymbol{\beta}, \boldsymbol{\gamma} \in \mathbb{R}^d$ and $\epsilon \in \mathbb{R}$.

N.B. t and s in (2) and in (3), resp, different [non-Minkowskian] "times".

Unification: Bargmann manifolds

A *Bargmann manifold* is

(i) a $(d+2)$-dim manif B

(ii) endowed with metric G of signature $(d+1,1)$

(iii) carries nowhere vanishing, complete, null "vertical" vector ξ, parallel-transported by Levi-Civita connection, ∇.

L. P. Eisenhart, "Dynamical trajectories and geodesics", Annals. Math. **30** 591-606 (1928).

C. Duval, G. Burdet, H. P. Kunzle and M. Perrin, "Bargmann Structures and Newton-Cartan Theory," Phys. Rev. D **31** (1985) 1841.

Flat Bargmann structure $\sim$ Minkowski space :

$$B = \mathbb{R}^d \times \mathbb{R} \times \mathbb{R} = \left\{ \begin{pmatrix} \boldsymbol{x} \\ t \\ s \end{pmatrix} \right\}, \tag{13}$$

$$G = \delta_{AB}\, dx^A dx^B + 2dtds, \tag{14}$$

$$\xi = \partial_s\,. \tag{15}$$

Both s & t light-cone (null), coords. t has dimension of time, coordinate s has that of action/mass.

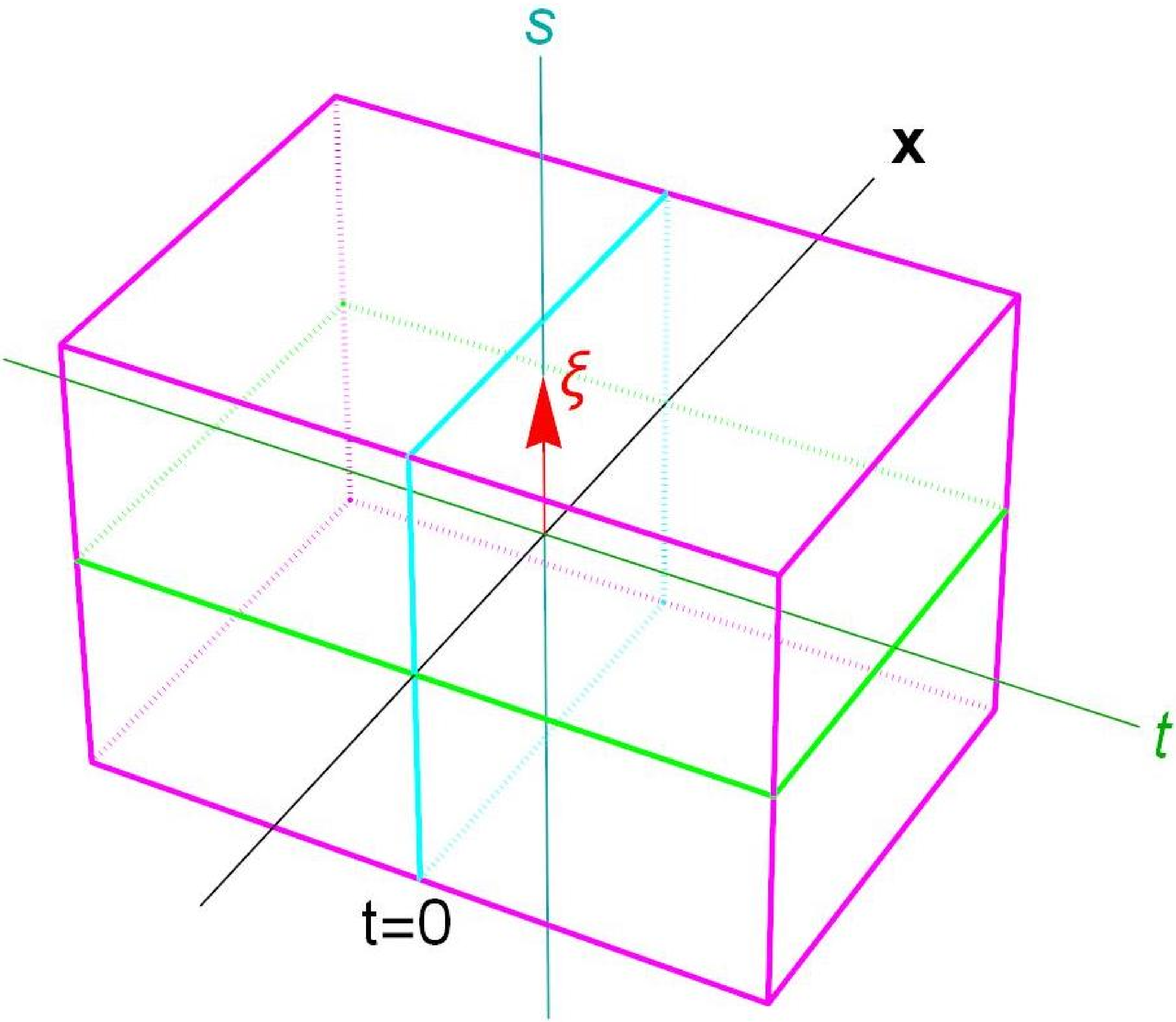


Fig. 4 : Bargmann space : $(d+1,1)$ dim manifold with Lorentz metric & coordinates $(\boldsymbol{x}, t, s)$, endowed with covariantly constant null vector $\xi = \partial_s$.

• Factoring out "vertical" translations along ξ, $(d+1)$-dim quotient acquires Newton-Cartan structure

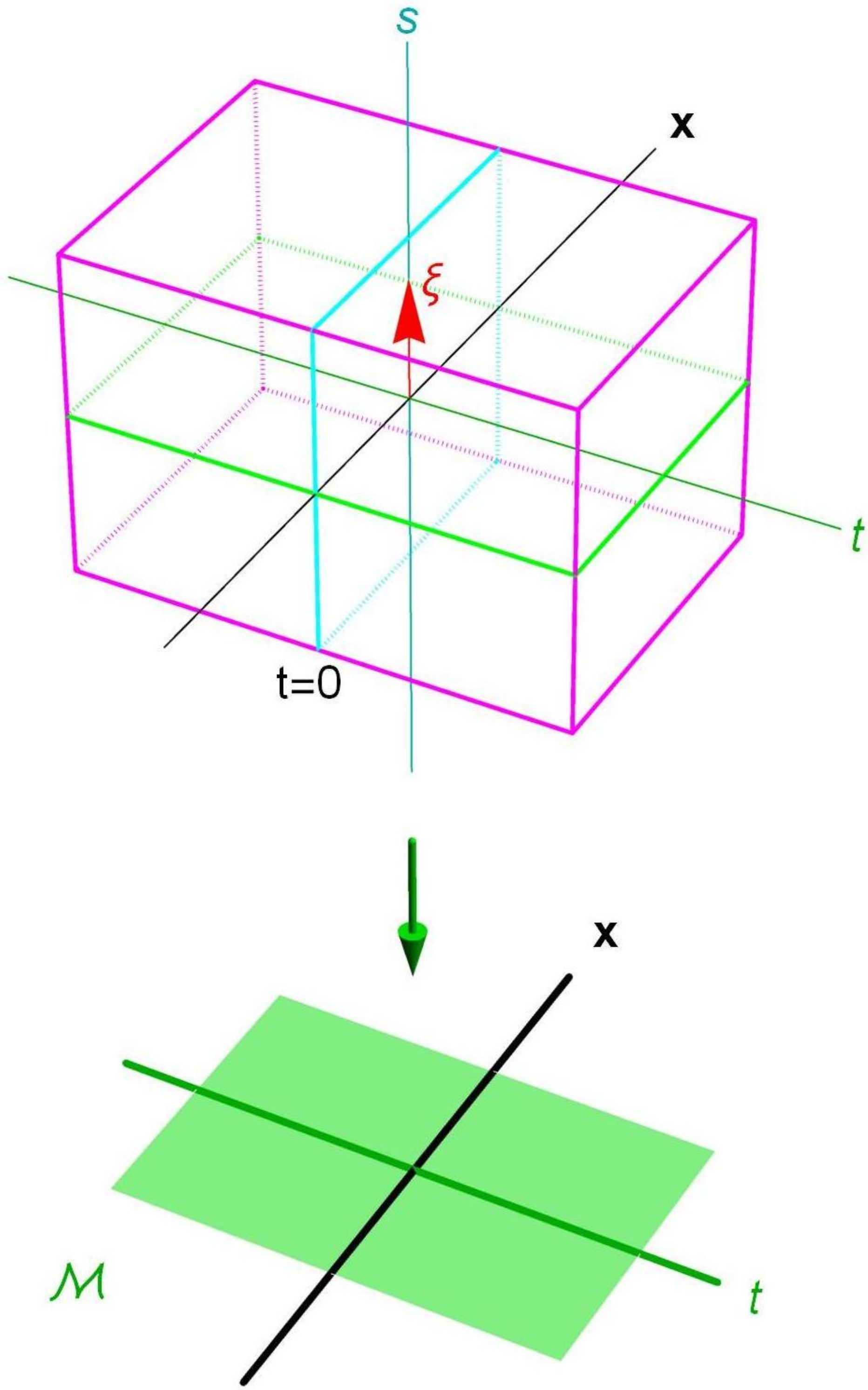


Fig. 5a : Bargmann projects to Galilean space-time.

• One-parameter family $C_t \subset B$ of $(d+1)$-dim sections $t = \text{const}$ turns off $dtds$ in metric (14), leaving singular "metric" $\delta_{AB}\, dx^A dx^B \rightsquigarrow C_t$ admits flat Carroll structure (*same* for all $t \in \mathbb{R}$) embedded into Bargmann.

$$\text{for} \quad t = 0: \qquad \mathcal{C}_0 = \left\{ \begin{pmatrix} \boldsymbol{x} \\ 0 \\ s \end{pmatrix} \right\} \tag{16}$$

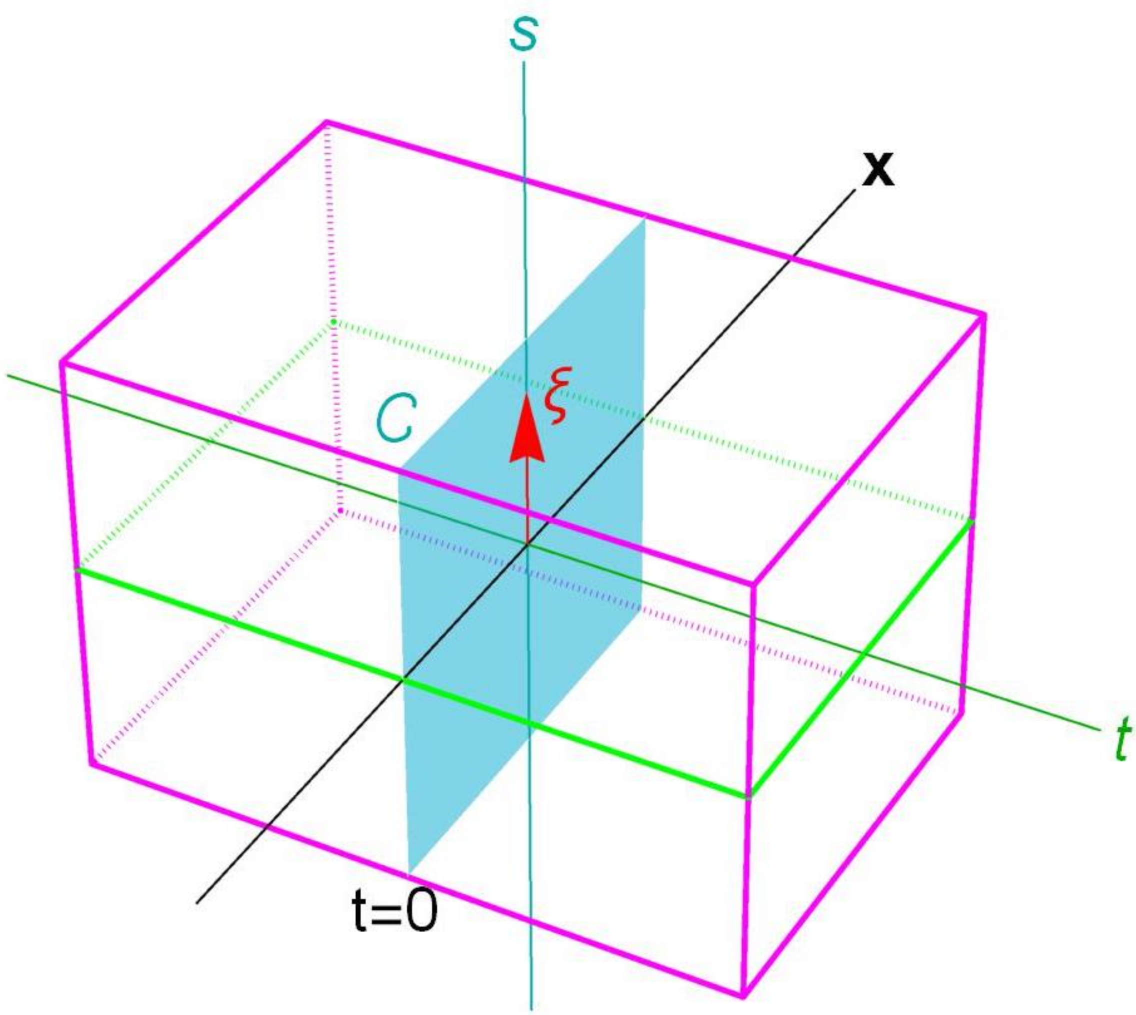


Fig.5b : $t = \text{const}$ slice is "Carroll space-time" $\mathcal{C}$ embedded into Bargmann.

Symmetries

ξ-preserving isometries of Bargmann :

$$a = \begin{pmatrix} R & \mathbf{b} & 0 & \mathbf{c} \\ 0 & 1 & 0 & e \\ -\mathbf{b}^T R & -\frac{1}{2}\mathbf{b}^2 & 1 & f \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad (17)$$

where $R \in \mathrm{O}(d)$, $\mathbf{b}, \mathbf{c} \in \mathbb{R}^d$, and $e, f \in \mathbb{R}$ form centrally extended Galilei [$\equiv$ Bargmann] group **Barg** $\equiv \mathrm{Barg}(d+1)$. Boost :

$$\begin{pmatrix} \boldsymbol{x} \\ t \\ s \end{pmatrix} \to \begin{pmatrix} \boldsymbol{x} + \mathbf{b}t \\ t \\ s - \mathbf{b}\cdot\boldsymbol{x} - \frac{1}{2}\mathbf{b}^2 t \end{pmatrix} \qquad (18)$$

N.B. : lifting ordinary wave fct to equivariant ($\equiv \partial_s \Psi = im\Psi$) on B-space, Galilei boost action (8) is Bargmann action. Affine action on $\begin{pmatrix} \boldsymbol{x} \\ t \\ s \\ 1 \end{pmatrix} \rightsquigarrow$ Bargmann algebra $\mathfrak{barg} \equiv \mathfrak{barg}(d+1)$

$$\boxed{(\omega^A_B\, x^B + \beta^A\, t + \gamma^A)\frac{\partial}{\partial x^A} + \varepsilon\frac{\partial}{\partial t} + (\varphi - \beta_A\, x^A)\,\frac{\partial}{\partial s}} \qquad (19)$$

where $\omega \in \mathfrak{so}(d)$, $\boldsymbol{\beta}$, $\boldsymbol{\gamma} \in \mathbb{R}^d$, $\varepsilon, \varphi \in \mathbb{R}$.

Seen before: restriction of Bargmann space to $t = 0$ is Carroll manifold, left invariant by restriction of Bargmann action with $e = 0 \rightsquigarrow$ Carr embedded into Bargmann group,

$$\iota : \begin{pmatrix} R & 0 & \mathbf{c} \\ -\mathbf{b}^T R & 1 & f \\ 0 & 0 & 1 \end{pmatrix} \hookrightarrow \begin{pmatrix} R & \mathbf{b} & 0 & \mathbf{c} \\ 0 & 1 & 0 & e=0 \\ -\mathbf{b}^T R & -\frac{1}{2}\mathbf{b}^2 & 1 & f \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{20}$$

$R \in \mathrm{O}(d)$, $\mathbf{b}, \mathbf{c} \in \mathbb{R}^d$, $f \in \mathbb{R}$.

Carr$(d+1)$: $e = 0$ subgroup of Barg(d+1).

Infinitesimally:

$$(\omega^A_B\, x^B + \gamma^A)\frac{\partial}{\partial x^A} + (\varphi - \beta_A\, x^A)\,\frac{\partial}{\partial s} \tag{21}$$

$\omega \in \mathfrak{so}(d)$, $\boldsymbol{\beta}$, $\boldsymbol{\gamma} \in \mathbb{R}^d$, $\varphi \in \mathbb{R}$ (seen before).

N.B. : for $t = t_0$ Carroll boost acts as

$$v \to v - \mathbf{b}.\boldsymbol{x} - \frac{1}{2}\mathbf{b}^2\, t_0 \tag{22}$$

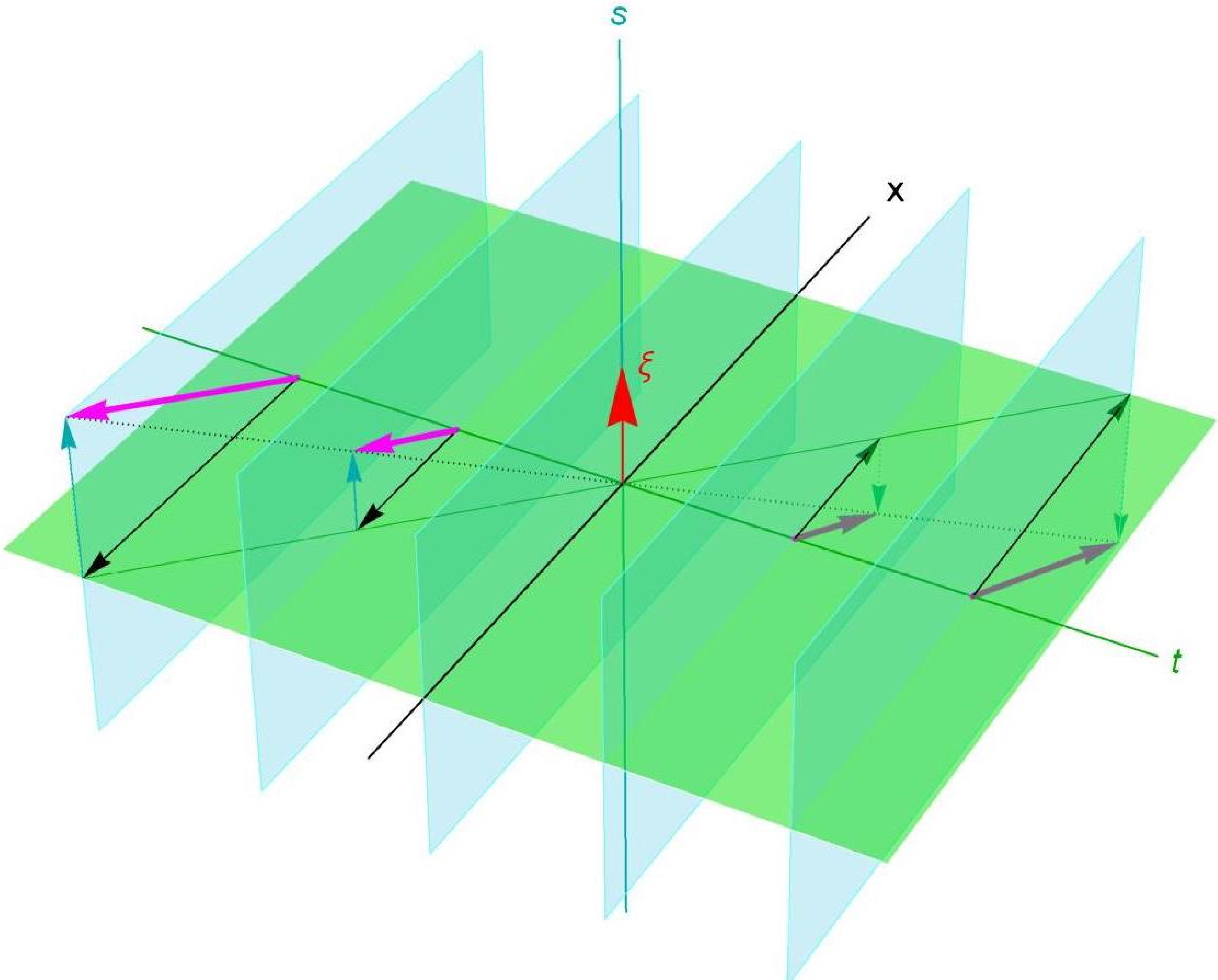


Fig.6 Boost acting on Bargmann space

Plane gravitational waves

$$ds^2 = d\boldsymbol{X}^2 + 2dUdV - K(U,\boldsymbol{X})\,dU^2 \qquad (23)$$

U and V light-cone coords, $\boldsymbol{X} = (X_1, X_2) \sim$ transverse plane. Brinkmann coordinates.

Only non-vanishing curvature components are $R^i_{UjU} = -R^V_{ijU} = -K_{ij} \quad \Rightarrow \quad$ vacuum Einstein eqn $\triangle K = 0$ satisfied with

$$K(U,\boldsymbol{X}) = \mathcal{A}(U)\Big(X_1^2 - X_2^2\Big) + \mathcal{B}(U)X_1X_2. \qquad (24)$$

Clue: (23) Bargmann space $\sim$ anisotropic oscillator.

Isometries : Bondi et al 1959. 5-parameters. Gibbons'75: 3 translations + 2 MYSTERIOUS !

1st step : BJR (Baldwin-Jeffery-Rosen) coord system, in which quadratic "scalar potential" term, KdU^2 in (23) is traded for "time"-dependent" transverse metric $a_{ij}(u)$, while leaving $U = u$ unchanged.

Achieved by solving Sturm-Liouville pb

$$\ddot{P}_{kj} = K_{kr} P_{rj} \quad \text{s.t.} \quad P^\dagger \dot{P} = \dot{P}^\dagger P. \tag{25}$$

for U-dept. 2×2 matrix $P_{kj} \equiv P_{kj}(U)$. Put

$$X^i = P_{ij}\, x^j \qquad U = u \tag{26a}$$

$$a_{ij}(u) = P_{ri} P_{rj}, \quad V = v - \frac{1}{4} \frac{da_{ij}}{du} x^i x^j \tag{26b}$$

allows to present metric (23) in form

$$\boxed{ds^2 = a_{ij}(u) dx^i dx^j + 2dudv} \tag{27}$$

Souriau 1973 isometries $u \to u$, completed with

$$\boxed{\boldsymbol{x} \to \boldsymbol{x} + H(u)\mathbf{b} + \mathbf{c},} \tag{28a}$$

$$\boxed{v \to v - \mathbf{b} \cdot \boldsymbol{x} - \tfrac{1}{2} \mathbf{b} \cdot H(u)\mathbf{b} + f} \tag{28b}$$

where $H = (H_{ij})$ is 2×2 matrix

$$\boxed{H(u) = \int_{u_0}^{u} a^{-1}(w) dw.} \tag{29}$$

$\mathbf{c} \in \mathbb{R}^2 \sim$ transverse-space transl, $f \sim$ null transl along v coord. $\rightsquigarrow$ Carroll group.

Flat case: $a_{ij} = \delta_{ij} \Rightarrow H(u) = (u - u_0)\,\mathrm{Id}$

$$\boldsymbol{x} \to \boldsymbol{x} + u\,\mathbf{b}, \tag{30a}$$

$$u \to u, \tag{30b}$$

$$v \to v - \mathbf{b}\cdot\boldsymbol{x} - \tfrac{1}{2}\mathbf{b}^2\,u \tag{30c}$$

Usual boosts lifted to flat Bargmann space.

Conserved quantities for geodesic motion

$$\mathbf{p} = a\,\dot{\boldsymbol{x}}, \qquad \mathbf{k} = \boldsymbol{x}(u) - H(u)\,\mathbf{p}, \qquad m = 1, \tag{31}$$

linear momentum, boost-momentum & mass (choosen unity in our parameterization). Allows integrate eqns motion:

$$\boldsymbol{x}(u) = H(u)\,\mathbf{p} + \mathbf{k}, \tag{32a}$$

$$v(u) = -\tfrac{1}{2}\mathbf{p}\cdot H(u)\,\mathbf{p} + c\,u + v_0, \tag{32b}$$

where $c = \frac{1}{2}\mathrm{g}_{\mu\nu}\,\dot{x}^\mu\dot{x}^\nu$ is (constant) kinetic energy; $c : -/0/+$ if geodesic space/null/timelike. v_0 integration const.

In flat Minkowski space recover free motion

$$\boldsymbol{x}(u) = u\,\mathbf{p} + \mathbf{k}, \tag{33a}$$

$$v(y) = -\tfrac{1}{2}\mathbf{p}^2 u + c\,u + v_0, \tag{33b}$$

G. W. Gibbons and S. W. Hawking "Theory of the detection of short bursts of gravitational radiation," Phys. Rev. D **4** (1971) 2191.

What happens to detectors originally at rest after "sudden gravitational burst" ?

In Brinkmann coords : wave profile $K(\boldsymbol{X}, U) = \mathcal{A}(U)\Big((X^1)^2 - (X^2)^2\Big)$. Geodesic eqns

$$\frac{d^2X^1(U)}{dU^2} - K(U)X^1(U) = 0, \qquad (34a)$$

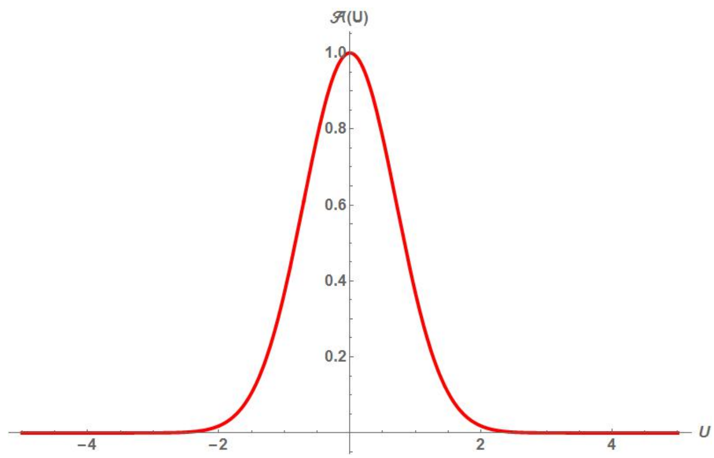
$$\frac{d^2X^2(U)}{dU^2} + K(U)X^2(U) = 0, \qquad (34b)$$

$$\frac{d^2V}{dU^2} + \frac{1}{2}K'(U)(X^2 - Y^2) + 2K(U)X(U)X'(U) + 2K(U)Y(U)Y'(U) = 0 \qquad (34c)$$

N.B. Eqns decoupled. Transverse motion same for all c (timelike/lightlike/spacelike).

• Toy model : burst $\sim$ Gaussian

$$\mathcal{A}(U) = \frac{1}{2} \exp[-U^2] \qquad (35)$$



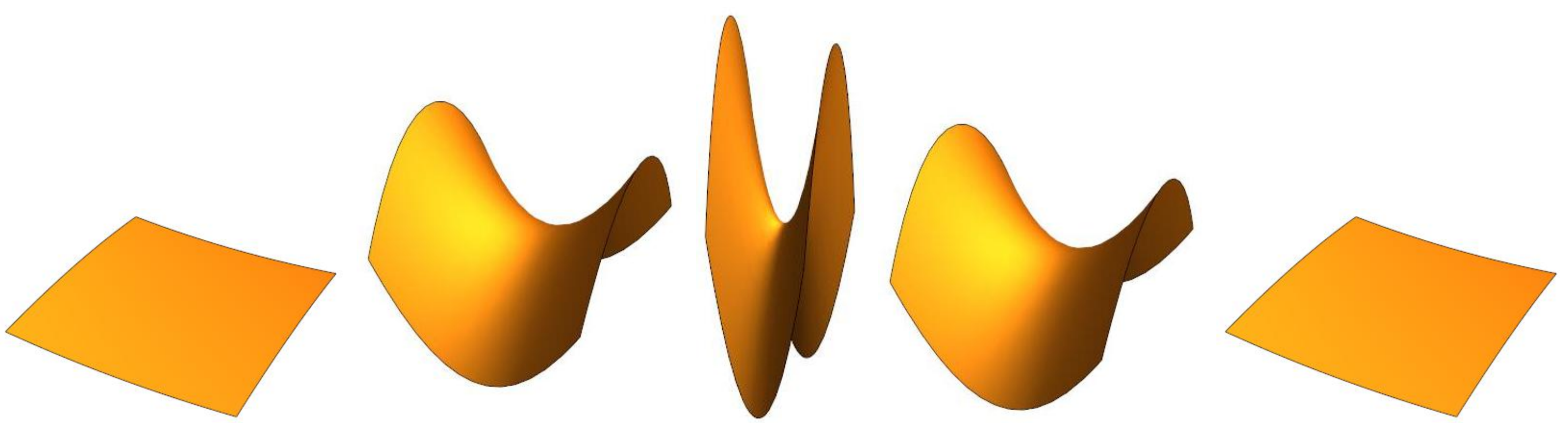


Fig.7a Wave profile of Gaussian burst.
(Gauss0-Movie)

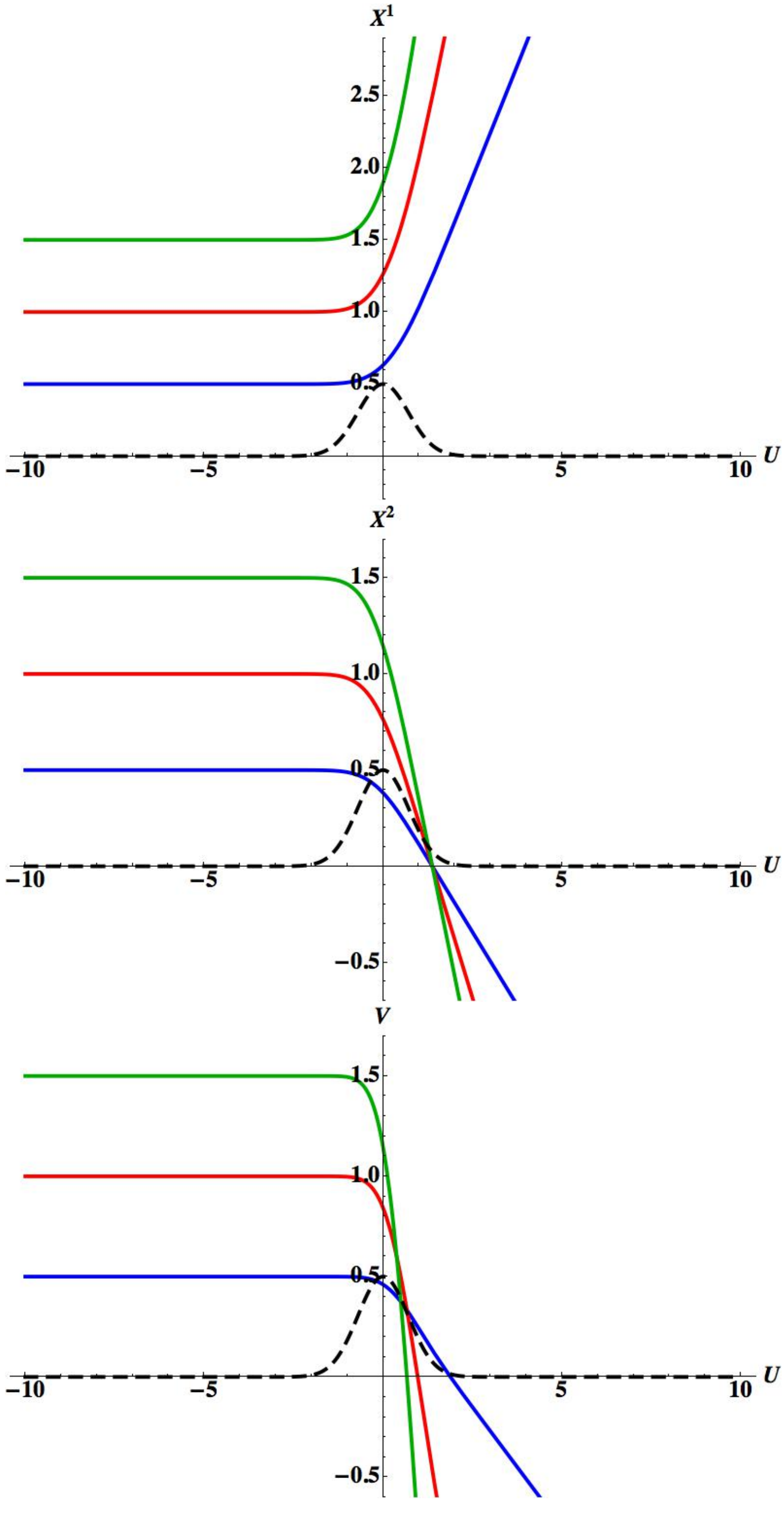


Fig.7b [Lightlike] Geodesics for Gaussian burst

Gibbons-Hawking 1971 $-$erfc $\sim$ collapse. Force $\sim$ (quadrupole momentum)$^{(iv)}$ $\Rightarrow$

$$\boxed{\mathcal{A}(U) = \frac{1}{2}\,\frac{d^3}{dU^3}\Big[\exp[-U^2]\Big]} \qquad (36)$$

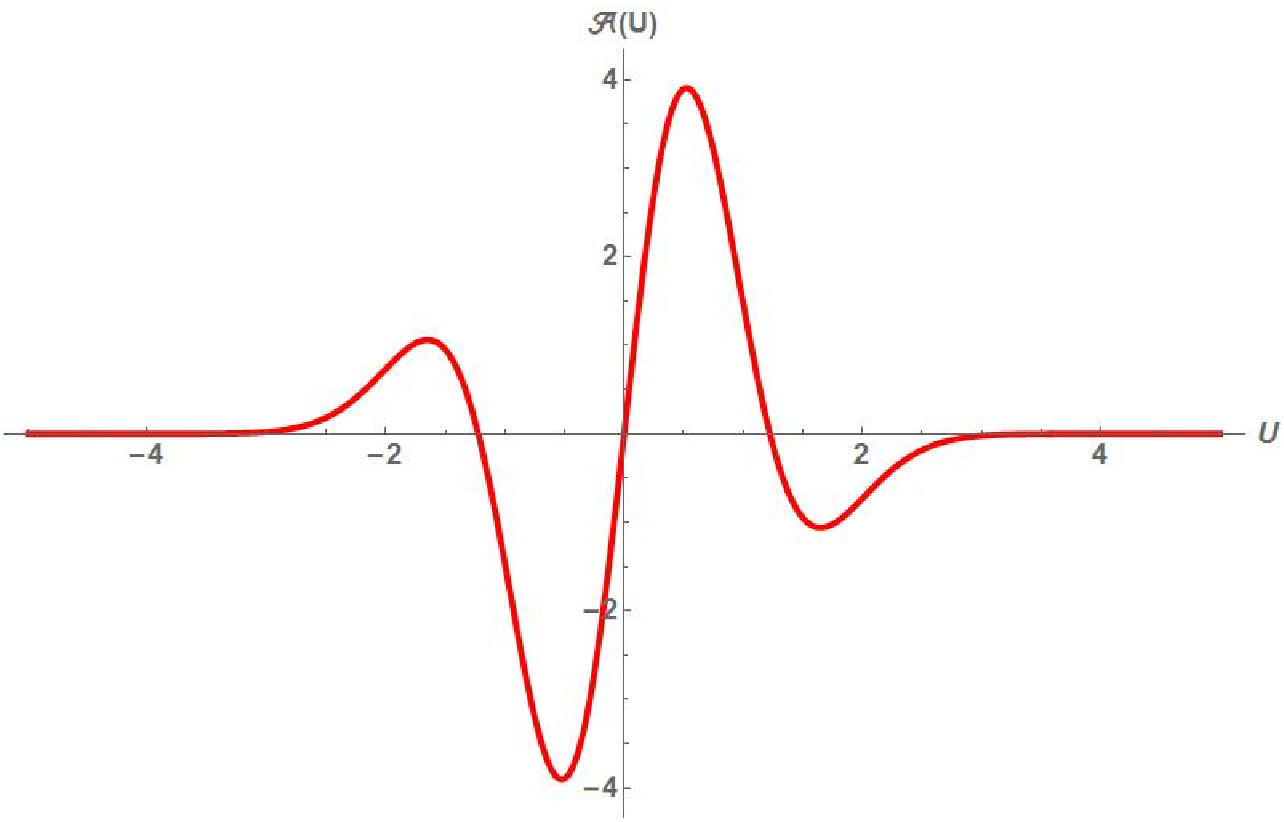


Fig.8 Wave profile of "sudden burst" $\mathcal{A}(U) = (\exp[-U^2])'''$ for $-5 < U < 5$.

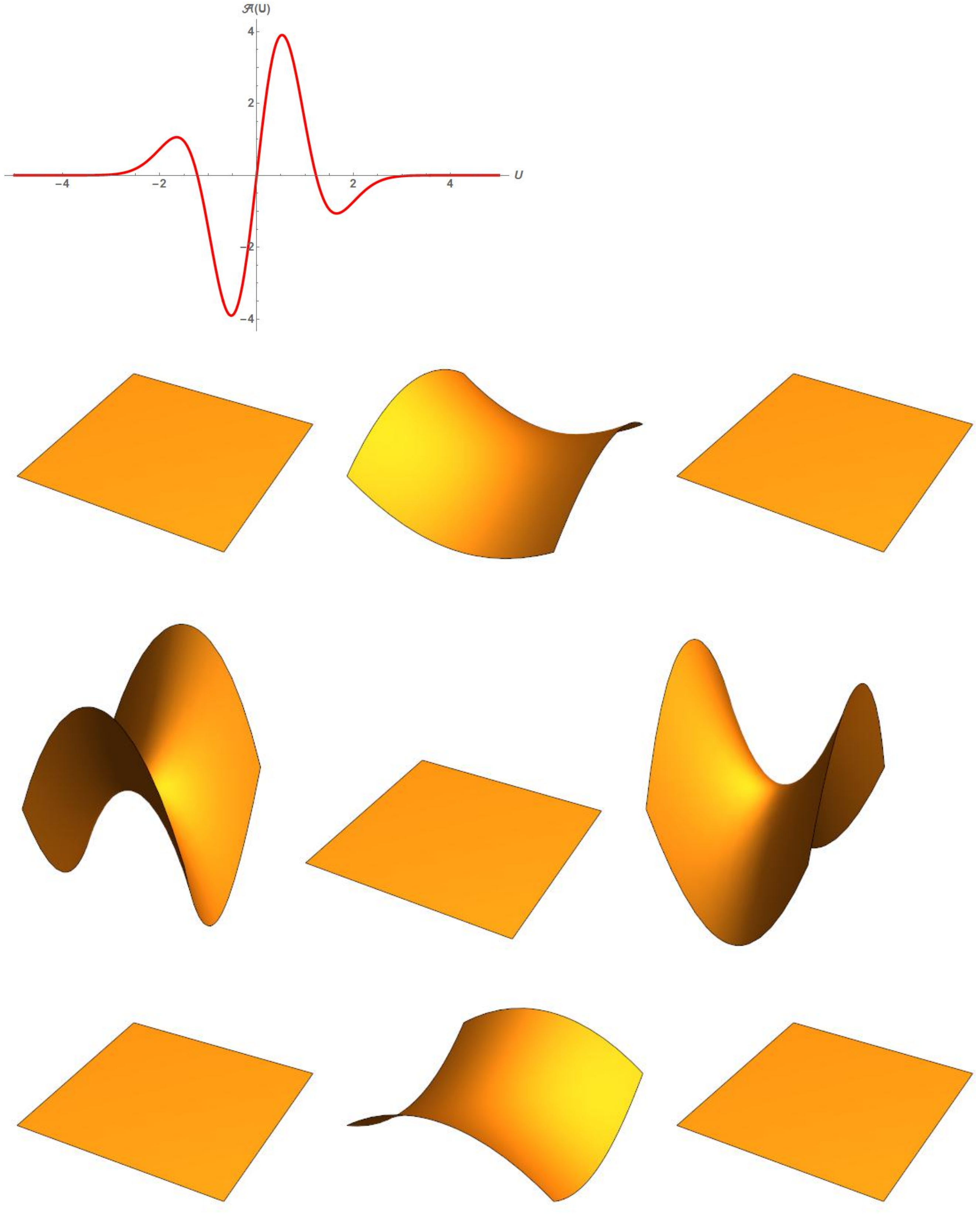
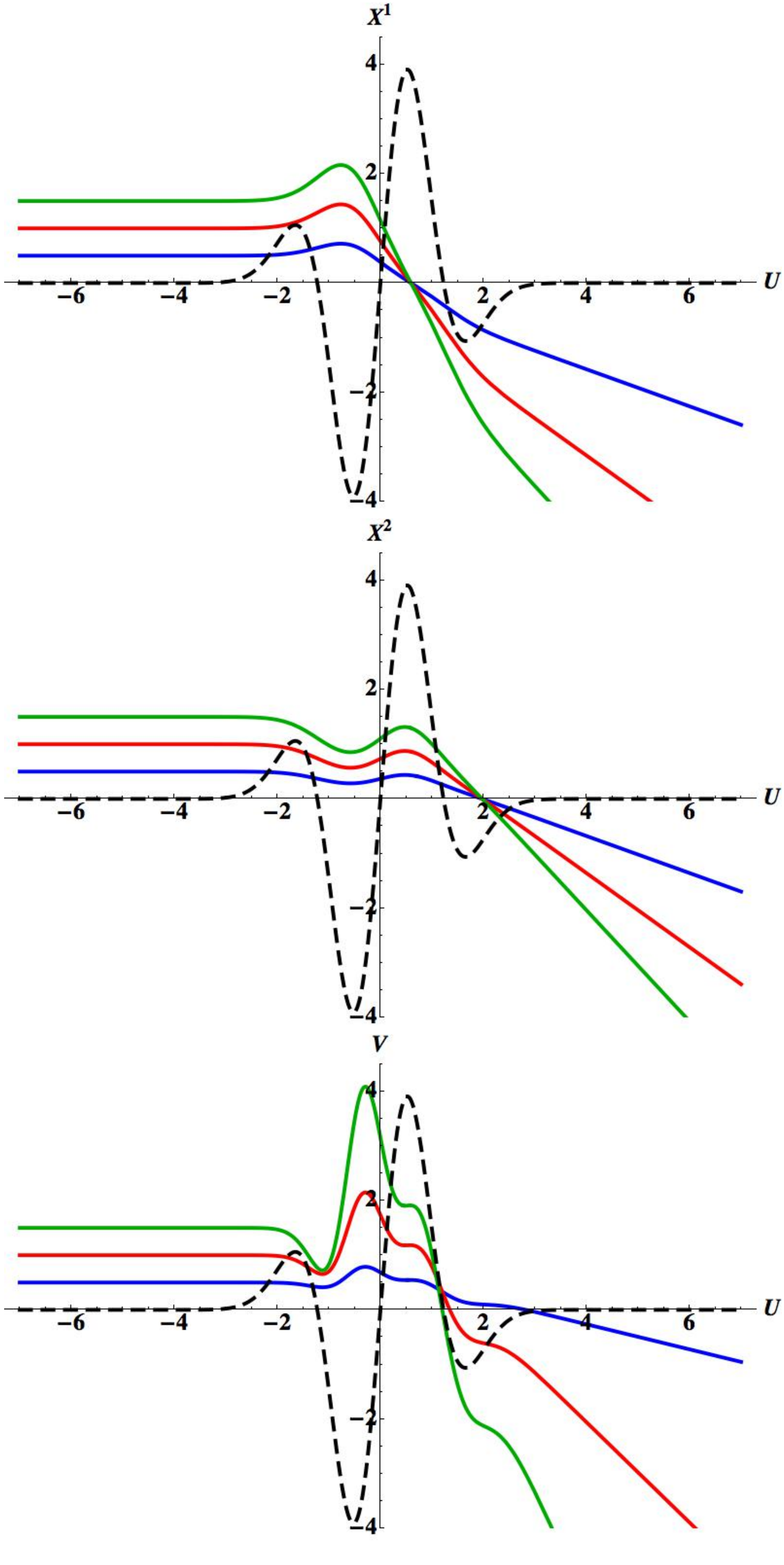


Fig.9 Geodesics at rest for $\mathcal{A}(U) = (\exp[-U^2])'''$ for $u << 0$

$x_0 = y_0 = v_0 = .5, 1, 1.5\,.$ (Gauss3-movie)


X^1
4
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4
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U
X^2
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−4
−2
2
4
6
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V
4
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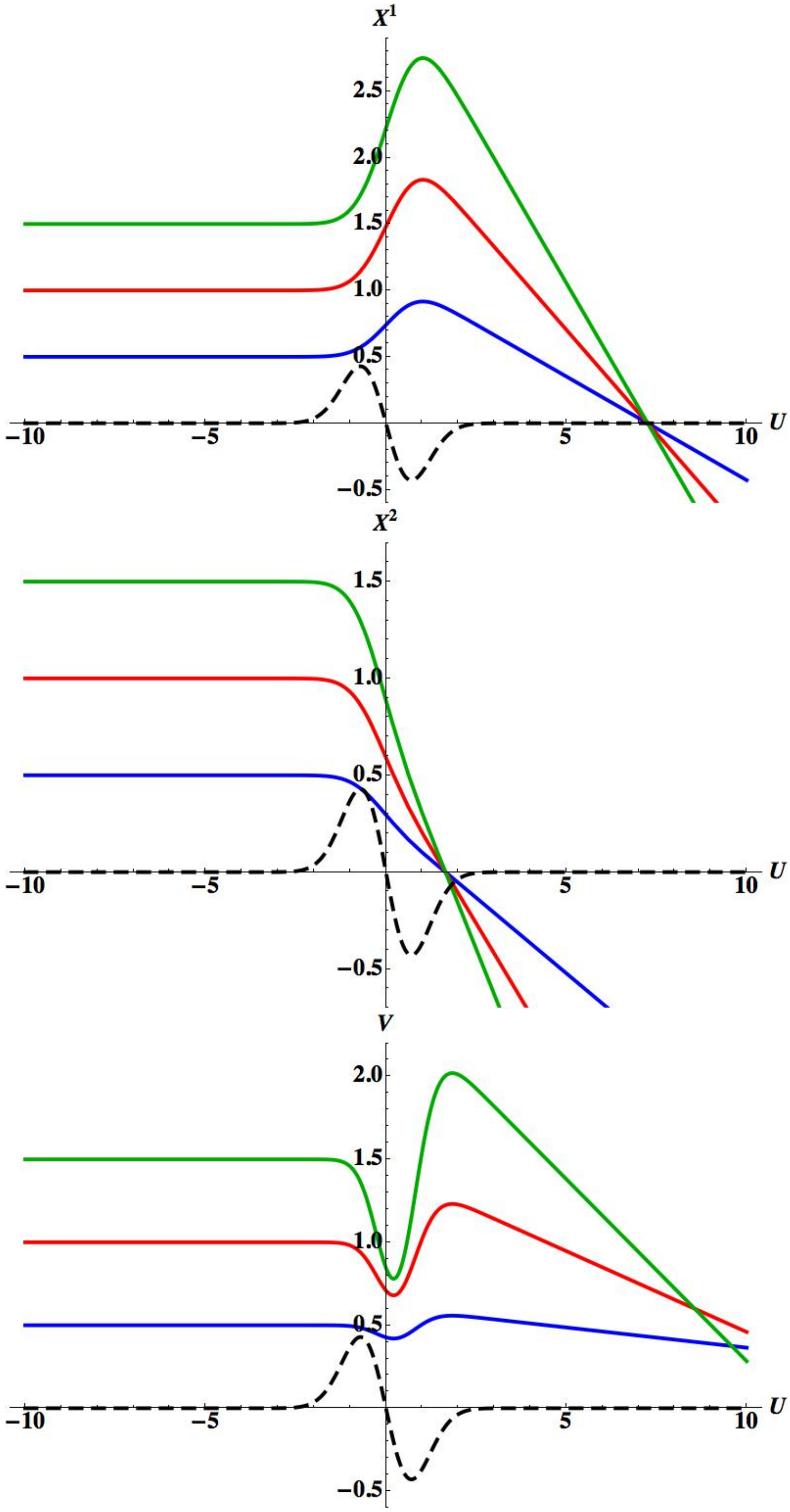
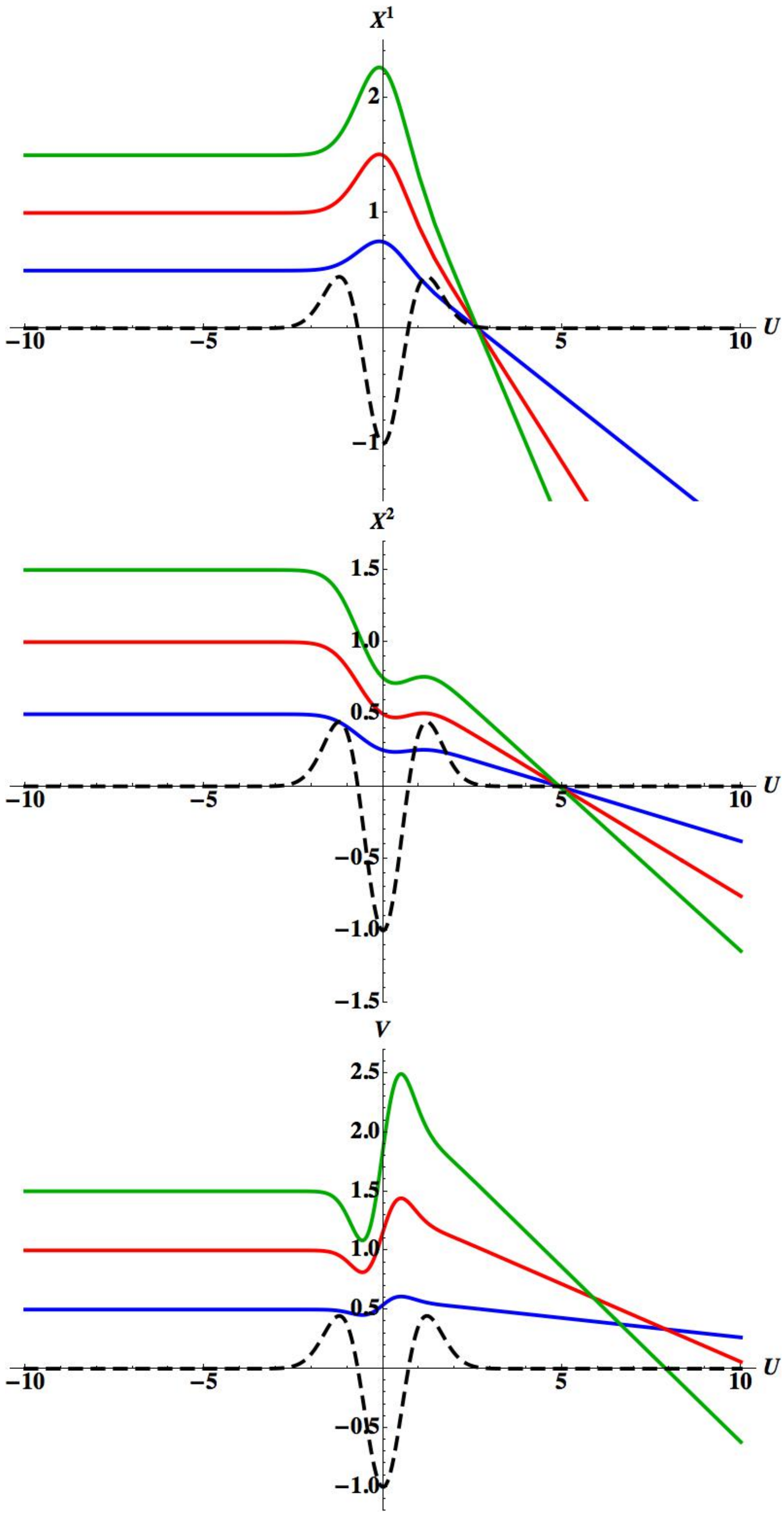
flyby : $\mathcal{A}(U) = \frac{1}{2}\frac{d}{dU}\left[\exp[-U^2]\right]$



Braginsky and Thorne (1987) Nature **327** 123 (1987)

$$\mathcal{A}(U) = \frac{1}{2}\frac{d^2}{dU^2}\left[\exp[-U^2]\right] \qquad (37)$$



Conclusion

On 2 July 1830 Jacobi

wrote to Legendre :

"M. Fourier avait l'opinion que le but principal des mathématiques était l'utilité publique et l'explication des phénomènes naturels ; mais un philosophe comme lui aurait dû savoir que le but unique de la science, c'est

l'honneur de l'esprit humain

et que sous ce titre, une question de nombres vaut autant qu'une question du système du monde."