The Schrödinger-Virasoro group: Geometry and Physics

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GARYFEST

"*Gravitation, Solitons and Symmetries*"

Le Studium & Université de Tours

March 22–34, 2017

Schrödinger & Virasoro

Figure: E. Schrödinger (1933)

Figure: M. A. Virasoro (recent)

Celebration of Gary's Honoris Causa degree, Tours 2017

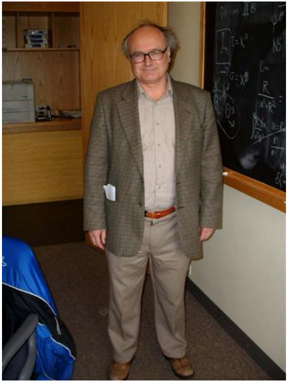

Figure: `GWG (Wikipedia)`

Figure: `HGD (Tours,2014)`

My conference talk is dedicated to Gary Gibbons, "Doctor Honoris Causa" of the Université de Tours. The work presented here is greatly inspired by his article "Dark Energy and the Schwarzian derivative" (2014).

The Schrödinger-Virasoro Lie algebra [Henkel, Roger-Unterberger]

- The Schrödinger-Virasoro (SV) algebra, $\mathfrak{sv}(\mathrm{d})$, introduced and named by **Henkel** in 1994 can be presented as the Lie algebra of vector fields

$$
\begin{aligned}
X_f &= f(t)\frac{\partial}{\partial t} + \frac{1}{2}f'(t)\,x^i\frac{\partial}{\partial x^i} - \frac{x_i x^i}{4}f''(t)\frac{\partial}{\partial s} \\
X_{\mathbf{a}} &= a^i(t)\frac{\partial}{\partial x^i} - x^i a_i'(t)\frac{\partial}{\partial s}, \\
X_\Omega &= \Omega^{ij}\left(x_i\frac{\partial}{\partial x^j} - x_j\frac{\partial}{\partial x^i}\right) \\
X_b &= b(t)\frac{\partial}{\partial s}
\end{aligned}
$$

  with f, a^i, b smooth functions of the time axis, $\mathbb{T}$, and $\Omega \in \mathfrak{so}(\mathrm{d})$.
- The SV algebra is an infinite-dimensional overalgebra of (i) the Schrödinger Lie algebra and (ii) the (centerless) Virasoro algebra, designed to extend the CFT of statistical mechanics to, e.g., symmetries of out of equilibrium systems.

Outline of the talk

1. Devise geometric origin of the Schrödinger-Virasoro group, $\mathrm{SV}(\mathrm{d})$, via **generalized conformal Bargmann structures**
2. Reveal SV as symmetry group of the space of
   - Schrödinger (S) operators
   - Lévy-Leblond (LL) operator

The The Schrödinger group, $\mathrm{Sch}(\mathrm{d})$

- The free Schrödinger equation
- The irreducible unitary representation of $\mathrm{Sch}(\mathrm{d})$ of mass m

The Schrödinger group, $\mathrm{Sch}(\mathrm{d})$, is the maximal group of (projective) Lie symmetries of the free Schrödinger equation of mass m, namely

$$\left(i\hbar\frac{\partial}{\partial t}+\frac{\hbar^2}{2m}\Delta_{\mathbb{R}^\mathrm{d}}\right)\psi(t,\mathbf{x})=0 \tag{1}$$

on Galilei spacetime $N=\mathbb{T}\times\mathbb{R}^\mathrm{d}$. Its unitary representation, ϱ, on the space of solutions $\psi_t\in L^2(\mathbb{R}^\mathrm{d},dx^1\ldots dx^\mathrm{d})$ of (1) is given by

$$\begin{aligned}\left[\varrho(\Phi)\psi\right](t,\mathbf{x}) \quad = \quad & e^{\frac{im}{\hbar}\left(-\frac{1}{2}\frac{f\|\mathbf{x}\|^2}{-ft+d}+\frac{\langle\mathbf{b},\mathbf{x}-\mathbf{b}(gt-e)\rangle}{-ft+d}-\langle\mathbf{b},\mathbf{c}\rangle+\frac{1}{2}\|\mathbf{b}\|^2\frac{gt-e}{-ft+d}-h\right)}\times \\ & \frac{1}{(-ft+d)^{\frac{\mathrm{d}}{2}}}\,\psi\left(\frac{gt-e}{-ft+d},A^{-1}\left[\frac{\mathbf{x}-\mathbf{b}(gt-e)}{-ft+d}-\mathbf{c}\right]\right)\end{aligned} \tag{2}$$

for all $\Phi=(A,\mathbf{b},\mathbf{c},d,e,f,g,h)\in\mathrm{Sch}(\mathrm{d})$, where $A\in\mathrm{SO}(\mathrm{d})$; $\mathbf{b},\mathbf{c}\in\mathbb{R}^\mathrm{d}$; $d,e,f,g,h\in\mathbb{R}$ with $dg-ef=1$.

We have the short exact sequence

$$0 \to \mathbb{R} \to \mathrm{Sch}(\mathrm{d}) \to \mathrm{Sch}(\mathrm{d})/\mathbb{R} = (\mathrm{SO}(\mathrm{d}) \times \mathrm{SL}(2,\mathbb{R})) \ltimes (\mathbb{R}^{\mathrm{d}} \times \mathbb{R}^{\mathrm{d}}) \to 1$$

The centerless Schrödinger group is isomorphic to the group of matrices

$$\begin{pmatrix} A & \mathbf{b} & \mathbf{c} \\ 0 & d & e \\ 0 & f & g \end{pmatrix} \in \mathrm{Sch}(\mathrm{d})/\mathbb{R} \tag{3}$$

with $(A, \mathbf{b}, \mathbf{c}, d, e, f, g)$ as above. Its projective action on spacetime, N, reads

$$\begin{pmatrix} \mathbf{x} \\ t \\ 1 \end{pmatrix} \longmapsto \begin{pmatrix} A & \mathbf{b} & \mathbf{c} \\ 0 & d & e \\ 0 & f & g \end{pmatrix} \begin{pmatrix} \mathbf{x} \\ t \\ 1 \end{pmatrix} \sim \begin{pmatrix} \frac{A\mathbf{x}+\mathbf{b}t+\mathbf{c}}{ft+g} \\ \frac{dt+e}{ft+g} \\ 1 \end{pmatrix}$$

and descends as that of $\mathrm{PSL}(2,\mathbb{R})$ on the time axis $\mathbb{T} \cong S^1$. The subgroup of dilations reveals a "dynamical exponent" $z = 2$, viz.,

$$(t, \mathbf{x}) \mapsto (g^{-2}\, t, g^{-1}\, \mathbf{x}) \qquad \text{where} \qquad g \in \mathbb{R}^\times$$

indicating that time is dilated twice as much as space.

The The Virasoro group, Vir

- The Schwarzian derivative: a tribute to Lagrange
- The Bott-Thurston cocycle

A tribute to Lagrange

If ϕ is a conformal mapping of $\mathbb{C}$, Lagrange introduces the function

$$S(\phi) = -2\sqrt{\phi'}\left(\frac{1}{\sqrt{\phi'}}\right)''$$

in his treatise on the *cartes géographiques* — Vol IV des œuvres complètes, cf. [Guieu-Roger, Ovsienko-Tabachnikov].

- This Lagrangian is, today, called the Schwarzian (derivative)

$$\boxed{S(\phi) = \frac{\phi'''}{\phi'} - \frac{3}{2}\left(\frac{\phi''}{\phi'}\right)^2}$$

  of ϕ and is an object of projective geometry.
- It defines a non-trivial 1-cocycle, $\mathcal{S}$, of $\mathrm{Diff}_+(\mathbb{T})$ with coefficients in the module of quadratic differentials $\mathcal{Q}(\mathbb{T})$:

$$\mathcal{S}(\phi_1 \circ \phi_2) = \phi_2^* \mathcal{S}(\phi_1) + \mathcal{S}(\phi_2)$$

  and has kernel $\mathrm{PSL}(2,\mathbb{R})$.

The Virasoro group

The Virasoro group, Vir, is the central extension of $\mathrm{Diff}_+(\mathbb{T})$, namely

$$0 \to \mathbb{R} \to \mathrm{Vir} \to \mathrm{Diff}_+(\mathbb{T}) \to 1$$

defined by the Bott-Thurston 2-cocycle

$$\mathbf{BT}(\phi_1, \phi_2) = -\frac{1}{2} \int_{\mathbb{T}} \mathcal{E}(\phi_1 \circ \phi_2) \mathcal{A}(\phi_2)$$

with $\mathcal{E}(\phi) = \log \phi'$ and $\mathcal{A}(\phi) = d\mathcal{E}(\phi)$, respectively the Euclidean and affine 1-cocycles of $\mathrm{Diff}_+(\mathbb{T})$.

The Schrödinger-Virasoro group & Bargmann structures

- The Galilei-Virasoro group and Galilei structures.
- Geometric definition of the Schrödinger-Virasoro (SV) group [Henkel, Roger-Unterberger] in terms of ξ-conformal Bargmann structures.
- Nontrivial cohomology classes of SV.

Galilei & Newton-Cartan structures

Definition [Cartan, . . . , Havas, Künzle, . . .]

A Galilei structure on a connected oriented $(\mathrm{d}+1)$-dimensional spacetime manifold N is a pair (γ, θ) where $\gamma \geq 0$ is symmetric 2-contravariant tensor field of rank d, and $\theta \in \Omega^1(N)$, the "clock", satisfies $d\theta = 0$ & spans $\ker(\gamma)$.

Locally, we have a fibration $\pi_0 : N \to \mathbb{T}$ over the absolute Time axis (we will consider $\mathbb{T} \cong S^1$):

$$\mathbb{T} = N / \ker(\theta)$$

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The following definition [Cartan, Trautman, Künzle, ...] has been devised for a geometric formulation of Newton gravitation theory akin to GR.

Definition

A Newton-Cartan (NC) structure is a quadruple (N,γ,θ,∇) where (N,γ,θ) is a Galilei structure, and ∇ a symmetric affine connection such that $\nabla\gamma = 0$ and $\nabla\theta = 0$.

Galilei-Virasoro group

The Galilei group, $\mathrm{Gal}(N, \gamma, \theta, \nabla)$, is the (finite-dimensional) group of automorphisms of $(N, \gamma, \theta, \nabla)$ [Trautman].

Two Galilei structures are **conformally related**, $(\gamma, \theta) \sim (\hat{\gamma}, \hat{\theta})$, iff $\hat{\gamma} = \lambda^{-1}\gamma$ and $\hat{\theta} = \lambda\theta$ with $\lambda \in C^\infty(N, \mathbb{R}_+^\times)$; hence $d\lambda \wedge \theta = 0$ or $\lambda \in \pi_0^* C^\infty(\mathbb{T}, \mathbb{R}_+^\times)$.

Definition

The Galilei-Virasoro group is the group, $\mathrm{GV}(N, \gamma, \theta)$, of orientation preserving **automorphisms** of the conformal Galilei structure $(N, \gamma \otimes \theta)$.

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Remark [D-Burdet-Perrin]

The (centerless) Schrödinger group of a NC manifold is the group

$$\text{Sch}(N, \gamma, \theta, \nabla) = \text{GV}(N, \gamma, \theta) \cap \mathrm{Proj}(N, \nabla)$$

Here $\mathrm{Proj}(M, \nabla)$ is the group of projective transformations of ∇ (that permute geodesics up to reparametrization, i.e., preserve “free fall”).

▶ The GV group structure involves crucially densities. The Diff($\mathbb{T}$)-module, $\mathcal{F}_\delta(\mathbb{T})$, of δ-densities of $\mathbb{T}$ can be trivialized via a non-vanishing 1-form, e.g., as $\alpha = \alpha_0\,|dt|^\delta$ where $\alpha_0 \in C^\infty(\mathbb{T})$.
As in [Roger-Unterberger] we use the trivial Galilei structure

$$N = \mathbb{T}\times\mathbb{R}^{\mathrm{d}}, \quad \gamma = \delta^{ij}\partial_i\otimes\partial_j, \quad \theta = dt, \quad \mathrm{vol} = dt\wedge dx^1\wedge\cdots\wedge dx^{\mathrm{d}} \qquad (4)$$

to work out the group GV(d).

Proposition

Let $(N, \gamma\otimes\theta)$ be the trivial conformal Galilei spacetime (4), then $\Phi \in \mathrm{GV}(\mathrm{d})$ iff $\Phi = (\phi; R, \boldsymbol{\alpha})$ with $\phi \in \mathrm{Diff}_+(\mathbb{T})$, $R \in \mathcal{F}_0\otimes\mathrm{SO}(\mathrm{d})$ & $\boldsymbol{\alpha} \in \mathcal{F}_{-\frac{1}{2}}\otimes\mathbb{R}^{\mathrm{d}}$; the GV-action is given by

$$\Phi\begin{pmatrix} t \\ \mathbf{x}\end{pmatrix} = \begin{pmatrix} \phi(t) \\ \sqrt{\phi'(t)}R(\phi(t))\,\mathbf{x} + \boldsymbol{\alpha}(\phi(t))\end{pmatrix}$$

The group law can be computed, yielding the ∞-dimensional structure

$$\mathrm{GV}(\mathrm{d}) = \mathrm{Diff}_+(\mathbb{T}) \ltimes \left[(\mathcal{F}_0\otimes\mathrm{SO}(\mathrm{d})) \ltimes \left(\mathcal{F}_{-\frac{1}{2}}\otimes\mathbb{R}^{\mathrm{d}}\right)\right]$$

Bargmann structures: ambient description

Bargmann structures are natural extensions of **Newton-Cartan** (NC) structures. As shown by **Eisenhart** in 1929 the solutions of Newton equations of motion are in fact projections of null geodesics of a certain **Lorentz** manifold above spacetime. This was further elaborated as follows.

Definition-Theorem [D-Burdet-Künzle-Perrin]

1. A Bargmann manifold is a principal $(\mathbb{R},+)$-bundle
$$\pi : M \to N$$
with fundamental vector field ξ; the total space, M, is endowed with a metric, g, of signature $(\mathrm{d}+1,1)$ such that $g(\xi,\xi)=0$ and $\nabla\xi=0$.
2. The base of a Bargmann manifold (M,g,ξ) is a NC manifold $(N,\gamma,\theta,\nabla^N)$ with $\gamma=\pi_* g^{-1}$, $\xi_\flat=g(\xi)=\pi^*\theta$, and ∇^N the projection of the Levi-Civita connection, ∇, of (M,g). We will write $\Pi_\xi : g \mapsto (\gamma,\theta)$.

Lifting conformal Galilei structures

Proposition [D-Michel]

Let (N, γ, θ) be a Galilei spacetime and $\pi : M \to N$ a $(\mathbb{R}, +)$-fiber bundle with fundamental vector field ξ.

1. The Bargmann metrics $g \in \Pi_{\xi}^{-1}(\gamma \otimes \theta)$ are of the form

$$g = \lambda\Big(\pi^*\gamma^{-1} + 2\,\pi^*\theta \odot \omega\Big) \tag{5}$$

   where $\lambda \in C^\infty(M, \mathbb{R}_+^\times)$ & $d\lambda \wedge \xi_\flat = 0$, and ω is a principal connection.

2. If $g \in \Pi_{\xi}^{-1}(\gamma, \theta)$, and $\mathrm{d} = 1$, there exists a coordinate system (x, t, s) and a function $U \in C^\infty(N)$ sth $\gamma = \partial_x \otimes \partial_x$ & $\theta = dt$, with [Brinkmann]

$$g = dx^2 + 2dt\,ds - 2U(t, x)dt^2 \qquad \& \qquad \xi = \partial_s \tag{6}$$

The function U is Newtonian potential; null geodesics of (M, g) project as solutions of the equations of motion $\ddot{x} = -\partial_x U(t, x)$ [Eisenhart].

The Schrödinger-Virasoro group

We know that Bargmann metrics $g \in \Pi_\xi^{-1}(\gamma \otimes \theta)$ that project onto a given Galilei conformal structure are of the form (5). This conformal class being however too large, we propose the following

Definition [D-Michel]

We will call ξ-conformal class of g the class

$$[g]_\xi = \left\{ \lambda \left(g + \mu \, \xi_\flat \otimes \xi_\flat \right) \,\middle|\, \lambda, \mu \in C^\infty(M); \lambda > 0 \right\} \tag{7}$$

- NB The "conformal factor", λ, is (the pull-back of) a function of $\mathbb{T}$ since

$$d\lambda \wedge \xi_\flat = 0$$

[1] See [Bekenstein] for related notion in GR (so-called "disformal" rescalings of the metric).

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Definition [D-Michel]

We will call ξ-conformal class of g the class

$$[g]_\xi = \left\{\lambda\,(g + \mu\,\xi_\flat \otimes \xi_\flat)\;\middle|\;\lambda, \mu \in C^\infty(M);\, \lambda > 0\right\} \tag{7}$$

- NB The "conformal factor", λ, is (the pull-back of) a function of $\mathbb{T}$ since

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- This **extension**[1] of conformal rescalings is straightforward and natural since a Bargmann structure is given by a **pair** (g, ξ).

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Recalling that the group of automorphisms of our principal fibre bundle $\pi : M \to N$ is the group $\mathrm{Aut}(M, \xi) = \left\{\Phi \in \mathrm{Diff}(M) \,\middle|\, \Phi^*\xi = \xi\right\}$, we are now ready for the following

Definition [D-Michel]

The Schrödinger-Virasoro group of a Bargmann manifold is defined by

$$\boxed{\mathrm{SV}(M, g, \xi) = \left\{\Phi \in \mathrm{Aut}(M, \xi) \,\middle|\, \Phi^*[g]_\xi \subseteq [g]_\xi\right\}} \tag{8}$$

As an example, we will specialize this general definition to the canonical flat $(\mathrm{d}+1, 1)$-dimensional Bargmann structure and reveal the stucture of its Schrödinger-Virasoro group, $\mathrm{SV}(\mathrm{d})$, whose infinitesimal generators can be shown to span the above Lie algebra $\mathfrak{sv}(\mathrm{d})$.

Proposition

For the flat $(\mathrm{d}+1,1)$-dimensional Bargmann structure $M = (\mathbb{T}\times\mathbb{R}^{\mathrm{d}})\times\mathbb{R}$, $g = \eta = \delta_{ij}\,dx^i dx^j + 2dt\,ds$, $\xi = \partial_s$, we have the group isomorphism[a]

$$\boxed{\mathrm{SV(d)} \cong \mathrm{Diff}_+(\mathbb{T}) \ltimes \left[\mathrm{SO(d)} \ltimes \left((\mathcal{F}_{-\frac{1}{2}} \otimes \mathbb{R}^{\mathrm{d}}) \oplus_c \mathcal{F}_0\right)\right]} \qquad (9)$$

Putting $\Phi = (\phi; R, \alpha, \beta) \in \mathrm{SV(d)}$, and $\mathcal{A}(\phi) = \phi''/\phi'$, the SV action reads

$$\begin{pmatrix} t \\ \mathbf{x} \\ s \end{pmatrix} \overset{\Phi}{\mapsto} \begin{pmatrix} \phi(t) \\ \sqrt{\phi'(t)}\, R\mathbf{x} + \alpha(\phi(t)) \\ s - \frac{\|\mathbf{x}\|^2}{4}\mathcal{A}(\phi)(t) - \sqrt{\phi'(t)}\,\langle R\mathbf{x}, \alpha'(\phi(t))\rangle + \left(\beta - \frac{1}{2}\langle\alpha,\alpha'\rangle\right)(\phi(t)) \end{pmatrix}$$

[a] Here $[c] \in H^2\left(\mathcal{F}_{-\frac{1}{2}} \otimes \mathbb{R}^{\mathrm{d}}, \mathcal{F}_0\right)$ represented by $c(\alpha_1,\alpha_2) = \frac{1}{2}[\langle\alpha_1,\alpha_2'\rangle - \langle\alpha_2,\alpha_1'\rangle]$

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► This expands to SV original results of [Gibbons] related to $\mathrm{Diff}(\mathbb{R})$. The centrally-extended SV group is then plainly [Roger-Unterberger]

$$\widehat{\mathrm{SV}}(\mathrm{d}) \cong \mathrm{Vir} \ltimes \left[\mathrm{SO}(\mathrm{d}) \ltimes \left(\left(\mathcal{F}_{-\frac{1}{2}} \otimes \mathbb{R}^{\mathrm{d}}\right) \oplus_c \mathcal{F}_0\right)\right] \tag{10}$$

We exhibit non-trivial cohomology classes of SV generalizing the previous $\mathrm{Diff}_+(\mathbb{T})$ ones and associated with conformal Bargmann structures.

If $\Phi \in \mathrm{SV}(M, g, \xi)$ there exists $\lambda, \mu : \mathrm{SV} \to C^\infty(M, \mathbb{R})$ sth (Def. (8))

$$\boxed{\Phi^* g = \lambda(\Phi)\,(g + \mu(\Phi)\,\xi_\flat \otimes \xi_\flat) \qquad \& \qquad \Phi^* \xi = \xi} \tag{11}$$

Theorem [D-Michel]

The maps $\widetilde{\mathcal{E}} = \log(\lambda)$, resp. $\widetilde{\mathcal{A}} = d\widetilde{\mathcal{E}}$ & $\widetilde{\mathcal{S}} = \mu\,\xi_\flat$ are non-trivial 1-cocycles of $\mathrm{SV}(M, g, \xi)$ with values in $C^\infty(M, \mathbb{R})$, resp. $\Omega^1(M)$.

Proof (sketch): Suppose $\widetilde{\mathcal{E}}$ were a 1-coboundary; this would yield a $\mathrm{Diff}_+(\mathbb{T})$-invariant 1-form of $\mathbb{T}$. Idem for $\widetilde{\mathcal{A}}$. Also, if $\widetilde{\mathcal{S}}$ were a 1-coboundary, one would show that $\mathrm{SV} \subset \mathrm{Conf}(M, \hat{g})$ for some metric $\hat{g}$. ■

We exemplify this result in the trivial, flat, Bargmann case.

Theorem [D-Michel]

For the trivial $(\mathrm{d}+1,1)$-dimensional Bargmann structure, the previous SV 1-cocycles read respectively, for any $\Phi = (\phi; R, \alpha, \beta) \in \mathrm{SV}(\mathrm{d})$,

$$\begin{cases} \widetilde{\mathcal{E}}(\Phi) &= \mathcal{E}(\phi) \\ \widetilde{\mathcal{A}}(\Phi) &= d\widetilde{\mathcal{E}}(\Phi) \\ \widetilde{\mathcal{S}}(\Phi) &= \mu(\Phi)dt \end{cases}$$

where

$$\mu(\Phi) = -\left[\frac{\|\mathbf{x}\|^2}{2}\mathcal{S}(\phi) + 2(\phi')^{\frac{3}{2}}\langle R\mathbf{x}, \alpha''\rangle \circ \phi + \phi'(2\beta' - \langle \alpha, \alpha''\rangle) \circ \phi\right] \quad (12)$$

They extend to SV the Euclidean, affine and projective $\mathrm{Diff}_+(\mathbb{T})$-cocycles.

Action of the SV group on the space of Schrödinger and Lévy-Leblond operators

- The Schrödinger equation: ambient formulation.
- The SV group as a group of symmetries of the Schrödinger equation with arbitrary potential in the sense of [Niederer].
- The Lévy-Leblond equation: ambient formulation.
- The SV group as a group of symmetries of the Lévy-Leblond equation with arbitrary potential.

The Schrödinger equation: ambient formulation

The space of motions of a non-relativistic particle of mass m in a NC spacetime is symplectomorphic to $\mathcal{V}/\ker(\sigma)$ where

$$\mathcal{V}^{2n-2} = \left\{(x,p) \in T^*M \,\middle|\, g^{-1}(p,p) = 0; \, p \cdot \xi = m\right\}$$

is the "evolution space" endowed with its induced presymplectic 2-form, σ. To quantize the system, invoke Dirac's quantization of constraints & CEQ:

$$\mathcal{Q}_{w,w'}(g^{-1}(p,p)) = -\hbar^2 \Delta_g^Y \qquad \& \qquad \mathcal{Q}_{w,w'}(p \cdot \xi - m) = \frac{\hbar}{i} L_\xi - m$$

where $w = \frac{n-2}{2n}$ & $w' = \frac{n+2}{2n}$ are the Yamabe weights if $\dim M = n = \mathrm{d} + 2$.

Proposition [D-Burdet-Perrin, D-Gibbons-Horváthy, D-Lazzarini]

The coupled system of PDE on a Bargmann manifold (M, g, ξ), namely

$$\boxed{\Delta_g^Y \Psi = 0 \qquad \& \qquad \frac{\hbar}{i} L_\xi \Psi = m\Psi} \qquad (\Psi \in \mathcal{F}_w^{\mathbb{C}}(M))$$

descends as the Schrödinger equation on NC spacetime $(N, \gamma, \theta, \nabla^N)$.

Action of SV on ambient Schrödinger operator

Lemma [D-Michel]

Let $\hat{g} \in [g]_\xi$, i.e., $\hat{g} = \lambda(g + \mu\, \xi_\flat \otimes \xi_\flat)$ with $\lambda = \phi' > 0$, then

$$Ric_{\hat{g}} = Ric_g - \frac{1}{2}\Big[(n-2)\mathcal{S}(\phi) + \Delta_g \mu\Big]\xi_\flat \otimes \xi_\flat \quad (13)$$

$$\Delta^Y_{\hat{g}} = \Delta^Y_g - \mu\, L^2_\xi \quad (14)$$

Using the Diff(M)-naturality of Ric and Δ, then, for any $\Phi \in \mathrm{SV}(M, g, \xi)$, and wave-function $\Psi \in \mathcal{F}^{\mathbb{C}}_{\frac{n-2}{2n}}(M)$ we find

$$\begin{aligned} \Phi^*(\Delta^Y_g \Psi) &= \Delta^Y_{\Phi^* g} \Phi^* \Psi \\ &= \Delta^Y_{\hat{g}} \Phi^* \Psi \\ &= \left(\Delta^Y_g - \mu\, L^2_\xi\right) \Phi^* \Psi \\ &= \left[\Delta^Y_g + \frac{m^2}{\hbar^2}\mu(\Phi)\right] \Phi^* \Psi \end{aligned}$$

Since, $\Phi^*\xi = \xi$, we have $\Phi^*\left(\frac{\hbar}{i}L_\xi\Psi - m\Psi\right) = \frac{\hbar}{i}L_\xi\,\Phi^*\Psi - m\,\Phi^*\Psi$. Hence

Proposition

The action of the Schrödinger-Virasoro group on the space of Schrödinger operators reads as follows. For all $\Phi \in \mathrm{SV}(M, g, \xi)$:

$$\Phi^*\begin{pmatrix} \Delta_Y(\mathrm{g}) \\ \frac{\hbar}{i}L_\xi - m \end{pmatrix} = \begin{pmatrix} \Delta_Y(\mathrm{g}) - \mu(\Phi)(L_\xi)^2 \\ \frac{\hbar}{i}L_\xi - m \end{pmatrix} \tag{15}$$

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If Ψ is a solution of the Schrödinger equation of mass m on a Bargmann manifold, (M, g, ξ), then $\Phi^*\Psi$ is a solution of the Schrödinger equation with the supplemental potential

$$\boxed{U(\Phi) = -\frac{1}{2}\mu(\Phi)} \tag{16}$$

given by the **generalized Schwarzian derivative** (12).

The kernel of a group-cocycle being a subgroup, we have the following

Corollary

The subgroup of the Schrödinger-Virasoro group

$$\mathrm{Sch}(M,g,\xi) = \Big\{\Phi \in \mathrm{SV}(M,g,\xi) \,\Big|\, \mu(\Phi) = 0\Big\} \tag{17}$$

is the Schrödinger group. Its canonical unitary representation, ϱ, on the space of solutions of the Schrödinger equation, is given by

$$\varrho(\Phi)\Psi = \Phi_*\Psi \tag{18}$$

NB In the free case, the representation (18) of $\mathrm{SV}(\mathrm{d})$ is plainly given by (2).

The Lévy-Leblond equation: ambient formulation

The Lévy-Leblond (LL) equation has originally been devised in 1967 in order to reproduce, in the Galilean framework, Dirac's derivation of the celebrated spin-$\frac{1}{2}$ relativistic wave equation. We provide, here, an ambient formulation of this equation well-adapted to our treatment of the SV group.

- Assume that our Lorentzian manifold (M, g) admits a spin structure, i.e., a principal bundle $\mathrm{Spin}(M) \to M$ covering $2:1$ the bundle of its orthonormal frames. Let $S(M)$ be the associated spinor-bundle of M.
- The covariant derivative, ∇, of sections $\Psi \in \mathcal{S}(M) = \Gamma(M, S(M))$ reads: $\nabla_X \Psi = X^\alpha(\partial_\alpha \Psi + \lambda_\alpha \Psi)$ for all $X \in \mathrm{Vect}(M)$ with $\lambda_\alpha = \frac{1}{8}\left[\gamma^\beta, \partial_\alpha \gamma_\beta - \Gamma^\sigma_{\alpha\beta}\gamma_\sigma\right]$ where, for all $\alpha = 1, \ldots, n$, the (locally defined) gamma matrices γ_α generate the embedding of TM in the Clifford fiber-bundle of M.
- We will suppose n **odd** for simplicity to avoid dealing with chirality; see however [D-Horváthy-Palla] for the planar case $d = 2$.

Let us introduce the conformally-invariant Dirac operator

$$D_{\mathrm{g}} : \mathcal{S}(M) \otimes \mathcal{F}^{\mathbb{C}}_{w}(M) \to \mathcal{S}(M) \otimes \mathcal{F}^{\mathbb{C}}_{w'}(M)$$

of (M, g), where $w = \frac{n-1}{2n}$ & $w' = \frac{n+1}{2n}$ (indeed $D_{\hat{\mathrm{g}}} = D_{\mathrm{g}}$ for all $\hat{\mathrm{g}} \in [\mathrm{g}]$).

Proposition [D, D-Michel]

The coupled system of PDE on a Bargmann manifold (M, g, ξ), namely

$$\boxed{D_g \Psi = 0 \qquad \& \qquad \frac{\hbar}{i} L_\xi \Psi = m\Psi} \qquad (\Psi \in \mathcal{S}(M) \otimes \mathcal{F}^{\mathbb{C}}_{w}(M))$$

descends as the Lévy-Leblond equation on NC spacetime $(N, \gamma, \theta, \nabla^N)$.

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Example. Starting with a Bargmann triple $(\mathbb{R}^5, \mathrm{g}, \xi)$, with $\mathrm{g} \in [\eta]_\xi$, we do recover the **standard LL equation** in $(3+1)$-dimensional NC spacetime:

$$\begin{pmatrix} -i\sigma^j \partial_j & -\frac{2im}{\hbar} \\ \partial_t + \frac{i}{\hbar} V(t, \mathbf{x}) & i\sigma^j \partial_j \end{pmatrix} \begin{pmatrix} \psi'(t, \mathbf{x}) \\ \psi''(t, \mathbf{x}) \end{pmatrix} = 0$$

with arguments $\psi', \psi'' \in C^\infty(\mathbb{R}^4, \mathbb{C}^2)$, and scalar potential $V = -\frac{1}{2} m\mu$.

The spinorial Lie derivative

In the above LL equation, we have used the the "Lie derivative" of a spinor field $\Psi \in \mathcal{S}(M)$ with respect to $X \in \mathrm{Vect}(M)$ given by [Kosmann]

$$L_X \Psi = X^\alpha \nabla_\alpha \Psi - \frac{1}{4} \gamma^\alpha \gamma^\beta \nabla_{[\alpha} X_{\beta]} \Psi \qquad (19)$$

We know that the curvature

$$\Omega(X, Y) = L_X L_Y - L_Y L_X - L_{[X,Y]}$$

of this "Lie derivative" vanishes for all $X, Y \in \mathfrak{conf}(M, \mathrm{g})$ [Kosmann,Bourguignon].
We have the stronger result: $\Omega(X, Y) = 0$ for all $X, Y \in \mathfrak{sv}(M, \mathrm{g}, \xi)$.

Theorem [D-Michel]

The spinorial Lie derivative $L : \mathfrak{sv}(M, \mathrm{g}, \xi) \to \mathrm{End}(\mathcal{S}(M))$ given by (19) is a Lie algebra homomorphism.

Infinitesimal SV action on LL operators

With these preparations, we have the following

Proposition [D-Michel]

Given a spin Bargmann manifold (M, g, ξ), the infinitesimal action of the Schrödinger-Virasoro group on the space of LL operators reads

$$L_X \begin{pmatrix} D_{\mathrm{g}} \\ \frac{\hbar}{i} L_\xi - m \end{pmatrix} = \begin{pmatrix} -\frac{1}{2}\mathfrak{m}(X)\,\gamma(\xi)\nabla_\xi \\ 0 \end{pmatrix} \qquad (20)$$

for all $X \in \mathfrak{sv}(M, \mathrm{g}, \xi)$, where $\mathfrak{m}$ is the infinitesimal “Schwarzian cocycle”, i.e., $\mathfrak{m}(X) = \delta\mu(\Phi)|_{\{\delta\Phi = X, \Phi = \mathrm{Id}\}}$ (see (11)).

This confirms and generalizes a similar statement [Roger-Unterberger] worked out for the flat $(1+1)$-dimensional Galilei spacetime.

Conclusion & outlook

- We have realized the Schrödinger-Virasoro group of a Bargmann structure as $\mathrm{SV}(M,g,\xi) = \mathrm{Aut}(M,\xi) \cap \mathrm{Stab}([g]_\xi)$.
- The SV group is a group of invariance of the Schrödinger & LL eqs "up to a potential term", supplying generalized Schwarzian derivative.
- Replacing automorphisms of the principal bundle $M \to N$ by mere automorphisms of the **fibration**, we obtain $\overline{\mathrm{SV}}(M,g,\xi)$: extended Schrödinger-Virasoro group.
- The the group $\widetilde{\mathrm{SV}}$ of invariance of the Schrödinger-Newton equation [Diosi,Penrose] is such that $\mathrm{SV} \subset \widetilde{\mathrm{SV}} \subset \overline{\mathrm{SV}}$ [D-Michel]; see [Robertshaw-Tod].
- How to realize the **centrally extended** SV group, $\widehat{\mathrm{SV}}$, (via the BT cocycle) with some brand new extension of Bargmann structures?