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A TALE OF TWO DYONS

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Concerning the title:

After my abstract was submitted, I discovered that Gary had published in 1997 (with Neil Cornish) a paper with an almost similar title: "The tale of two centres".

Actually, when choosing this title I had in mind Edgar Poe's detective tales, particularly "The gold-bug", in which deciphering a secret message leads to a buried treasure...

4D Einstein-Maxwell: old-fashioned theory,
but still full of surprises!

– I will present a "new" exact stationary, asympt. flat solution of the EM equations, and the successive clues leading to its interpretation as a complex physical system of two extreme co-rotating dyonic NUTty black holes, held apart by an electrically charged rod which also acts as a Dirac-Misner string.

– Solution constructed 20 years ago (GC1997) using finite Geroch transformation.

– Restriction of EM4 to stationary solutions
(1 timelike Killing vector) $\longrightarrow$
reduction to 3D gravitating σ model
with "hidden" symmetry group $SU(2,1)$
of transformations between solutions
(e.g. Schwarzschild $\to$ RN-NUT).
Static and rotating solutions are not
related by $SU(2,1)$ transformations.

– Solutions with 2 commuting Killing vectors
(e.g. stationary axisymmetric):
combination of infinitesimal $SU(2,1)$ transf.
with infinitesimal linear transf. $K_i \to K_i'$
in the 2-Killing vector space
$\longrightarrow$ infinite dimensional Geroch group
$\Longrightarrow$ complete integrability.
Basis for inverse scattering transform methods.

• Finite Geroch transformation
(G. Clément, "From Schwarzschild to Kerr ..." Phys. Rev. D57 (1998) 4885, gr-qc/9710109):

– Problem: A linear transf. $\partial_t \to \partial'_t = \alpha\partial_t + \beta\partial_\varphi$ changes the asymptotic behaviour ("centrifugal force").

– Solution: Combine this with an $SU(2,1)$ transformation also changing the as. behaviour.

– Bertotti-Robinson solution to EM4: geometry $AdS_2 \times S^2$

$$ds^2 = -(x^2-1)dt^2 + \frac{dx^2}{x^2-1} + \frac{dy^2}{1-y^2} + (1-y^2)d\varphi^2$$

generated by a constant electric field $A = xdt$.

– This is related to Schwarzschild by an $SU(2,1)$ transf. $\Pi : \; S \to BR$

– The linear transformation $\mathcal{R}(\Omega, \gamma)$:
$d\varphi = d\varphi' - \Omega dt'$, $dt = \gamma dt'$ acting on BR does not change the as. behaviour ($x \to \infty$).

– The combined transformation

$$\Sigma = \Pi^{-1}\mathcal{R}(\Omega, \gamma)\Pi$$

transforms Schwarzschild $\longrightarrow$ Kerr.

– More generally, Σ acting on as. flat monopole stationary axisymm. solution of EM $\longrightarrow$ as. flat monopole + dipole solution.

• Axisymmetric stationary metric (Weyl):

$$ds^2 = -F(dt - \omega d\varphi)^2 + F^{-1}[e^{2k}(d\rho^2 + dz^2) + \rho^2 d\varphi^2].$$

Prolate spheroidal coords. (x, y):

$$\rho = \kappa(x^2 - 1)^{1/2}(1 - y^2)^{1/2}, \quad z = \kappa xy.$$

– Zipoy1966-Voorhees1970 vacuum metric:

$$F = \left(\frac{x-1}{x+1}\right)^{\delta}, \; e^{2k} = \left(\frac{x^2-1}{x^2-y^2}\right)^{\delta^2}, \; \omega = 0 \; (\delta \in \mathrm{R})$$

naked curvature singularity at $x = 1$,
except for $\delta = 1$ (Schwarzschild);
$\delta = 2$ solution first given by Darmois1927.

Rotating generalization: Tomimatsu+Sato1972 (δ integer). $\delta = 2$: naked ring singularity;
– Gibbons+Russel-Clark1973: TS2 has a causal boundary ($g_{\varphi\varphi} = 0$) and a non-curvature Misner-string singularity at $x = 1$;
– Kodama+Hikida2003: Two degenerate horizons at $x = \pm y = 1$, and a conical singularity at $x = 1$.

• Transformation Σ acting on ZV
$\rightarrow$ continuous family of rotating spacetimes with dipole electromagnetic field.

– $\delta = 1$: Kerr; $\delta = 1 + \epsilon$: "almost Kerr".

– Rotating solution for $\delta = 2$:
Ernst potentials:

$$\mathcal{E} = \frac{U - W}{U + W}, \quad \psi = \frac{V}{U + W}.$$

Kinnersley potentials:

$$U = p\frac{x^2 + 1}{2x} - iqy, \qquad (p = \sqrt{1 - q^2}, \;\; \varepsilon^2 = 1)$$

$$W = 1 + \frac{q^2}{2}\frac{1 - y^2}{x^2 - 1} + i\frac{pq}{2}\frac{y}{x}, \quad V = \varepsilon(1 - W).$$

– Simple enough, but for the explicit solution, must dualize (recover the vector potentials ω_i and A_i from the imaginary part of the Ernst scalar potentials $\mathcal{E}$ and ψ)!

– Black hole unicity theorems $\Longrightarrow$
this must be singular! . . . .

- The full solution $(x > 1,\ -1 \le y \le 1)$

$$ds^2 = -\frac{f}{\Sigma}\left(dt - \frac{\kappa\Pi}{f}d\varphi\right)^2 + \kappa^2\Sigma\left[e^{2\nu}\left(\frac{dx^2}{x^2-1} + \frac{dy^2}{1-y^2}\right) + f^{-1}(x^2-1)(1-y^2)d\varphi^2\right],$$

$$f = \frac{p^2(x^2-1)^2}{4x^2} - \frac{q^2x^2(1-y^2)}{x^2-1}$$

$$\Sigma = \left[\frac{px^2+2x+p}{2x} + \frac{q^2(1-y^2)}{2(x^2-1)}\right]^2 + q^2\left(\frac{p}{2x}-1\right)^2 y^2,$$

$$e^{2\nu} = \frac{4x^2(x^2-1)^2}{p^2(x^2-y^2)^3},$$

$$\Pi = \Pi_1(x)(1-y^2) + \Pi_2(x)(1-y^2)^2,$$

$$A = \frac{\varepsilon}{\Sigma}[\overline{v}dt + \kappa\Theta d\varphi],$$

$$\overline{v} = v_0(x) + v_1(x)(1-y^2) + v_2(x)(1-y^2)^2,$$

$$\Theta = \Theta_1(x)(1-y^2) + \Theta_2(x)(1-y^2)^2 + \Theta_3(x)(1-y^2)^3.$$

• Asymptotically flat metric.

– mass $M = 2\kappa/p$,

– angular momentum $J = -\kappa^2 q(4+p^2)/p^2$,

– dipole magnetic moment $\mu = \varepsilon\kappa^2 q$,

– quadrupole electric moment $Q_2 = \varepsilon\kappa^3 q^2/p$.

• Possible singularities:

– ring singularities ($\rho = \rho_0$)

$$\Sigma(x, y) = 0 \quad \text{(2 eqs.)}$$

no solution! ($\Sigma > (1+p)^2$)

– axial singularities ($\rho = 0$):

$$\begin{cases} \text{segment } R & (x = 1,\ -1 < y < 1), \\ \text{points} & (x = 1,\ y = \pm 1). \end{cases}$$

- Ergosurface

$$F \equiv f/\Sigma = 0.$$

2 components:

a) $f(x, y) = 0$, contains R ($f < 0$);

b) R itself ($\Sigma = \infty$).

- Causal boundary

$$g_{\varphi\varphi} \equiv F^{-1}\rho^2 - F\omega^2 = 0$$

contains R ($g_{\varphi\varphi} < 0$).

- Horizons

$$N^2 \equiv \rho^2/g_{\varphi\varphi} = 0 \text{ (with} g_{\varphi\varphi} > 0)$$

$\longrightarrow$ candidates $H_{\pm}(x = 1,\ y = \pm 1)$.

• Geodesics:
1st integral $\quad T+U=\epsilon \quad (\epsilon=-1,0,+1)$

$$T = \kappa^2 \Sigma e^{2\nu}\left(\frac{\dot{x}^2}{x^2-1}+\frac{\dot{y}^2}{1-y^2}\right) > 0,$$
$$U = \frac{(l-E\omega)^2 F}{\rho^2} - \frac{E^2}{F}.$$

• Near R $(x=1, y^2<1)$: $-F \propto \xi^2$, $\rho^2 \propto \xi^2$ $(\xi^2 \equiv x^2-1 \to 0)$

– $E \neq 0$: $U \gg \epsilon \Longrightarrow$ geodesics turn back before reaching R.

– $E=0$: geodesics terminate on R, but timelike or null geodesics ($\epsilon=-1$ or 0) cannot originate from ∞:
"harmless" naked singularity.

• Near $H_\pm$: Geodesics such that, near $x=1$,
$1-y^2 \sim X^2(x^2-1) \quad (X$ fixed)
can be continued through $x=\pm y=1$
to a region with $x<1$ and $y^2>1$
$\Longrightarrow$ 2 double horizons.

2 black holes $(H_\pm)$ held apart by a rod (R).

• Interlude: 2-black hole stationary solutions

• BPS superpositions
Majumdar,Papapetrou(1947): Static linear superpositions of N identical extremal BH.
Israel+Wilson,Perjès(1971): Stationary linear superpositions of N BH, with rod singularities (Hartle+Hawking,Bonnor+Ward(1972)).

• Weyl superpositions
Stationary axisym. linear superpositions of N identical non-extremal BH, with singular rods (Bach+Weyl1922, Israel+Khan1964).

• Extremal diholes
– Bonnor1966: Static solution, with only mass and magnetic dipole moment.
– Emparan2000: This is a dihole: 2 extreme magnetic RN BH, with equal masses and opposite magn. charges, held apart by a rod (can be replaced by an external magnetic field, at the expense of asymptotic flatness).

• Double Kerr

– Kramer+Neugebauer1980: Double Kerr-NUT.

– Bonnor+Steadman2003: For equal masses, this is as. flat if either the 2 spins are opposite, <u>or</u> the 2 spins are equal, and the rod between the holes is spinning

– Co-rotating double Kerr with massless, non-spinning rod (only conical singularity): Cabrera-Munguia et al., Manko+Ruiz(2017)

• Non-extremal diholes

– Emparan+Teo2001: Static non-extremal diholes with equal masses, opposite charges, and rod

– Generalization to 2 counter-rotating black dyons with opposite charges: Cabrera-Munguia et al.2013, Manko et al.2014

- The horizons: geometry

Blow up the horizons $x = \pm y = 1$ by transforming to the coords. (Kodama+Hikida):

$$X = \sqrt{\frac{1-y^2}{x^2-1}}, \quad Y = \frac{y}{x}.$$

On the horizons $Y = \pm 1$,

$$F_H = -\frac{q^2 X^2}{\Sigma_H(X)}, \quad \omega_H^{-1} = \Omega_H = -\frac{q}{\kappa\lambda(p)},$$

$$\Sigma_H(X) = \frac{p\lambda(p)}{2} + q^2(1+p)X^2 + \frac{q^4}{4}X^4.$$

– In the co-rotating near-horizon frame $\widehat{\varphi} = \varphi - \Omega_H t$,

$$ds^2_H = \frac{\kappa^2 \lambda(p)}{2pl(\theta)} \left[d\theta^2 + l^2(\theta) \sin^2 \theta d\widehat{\varphi}^2 \right],$$

$$l(\theta) \equiv \frac{p\lambda(p)}{2} \frac{(X^2+1)^2}{\Sigma_H(X)}, \quad X = \tan(\theta/2).$$

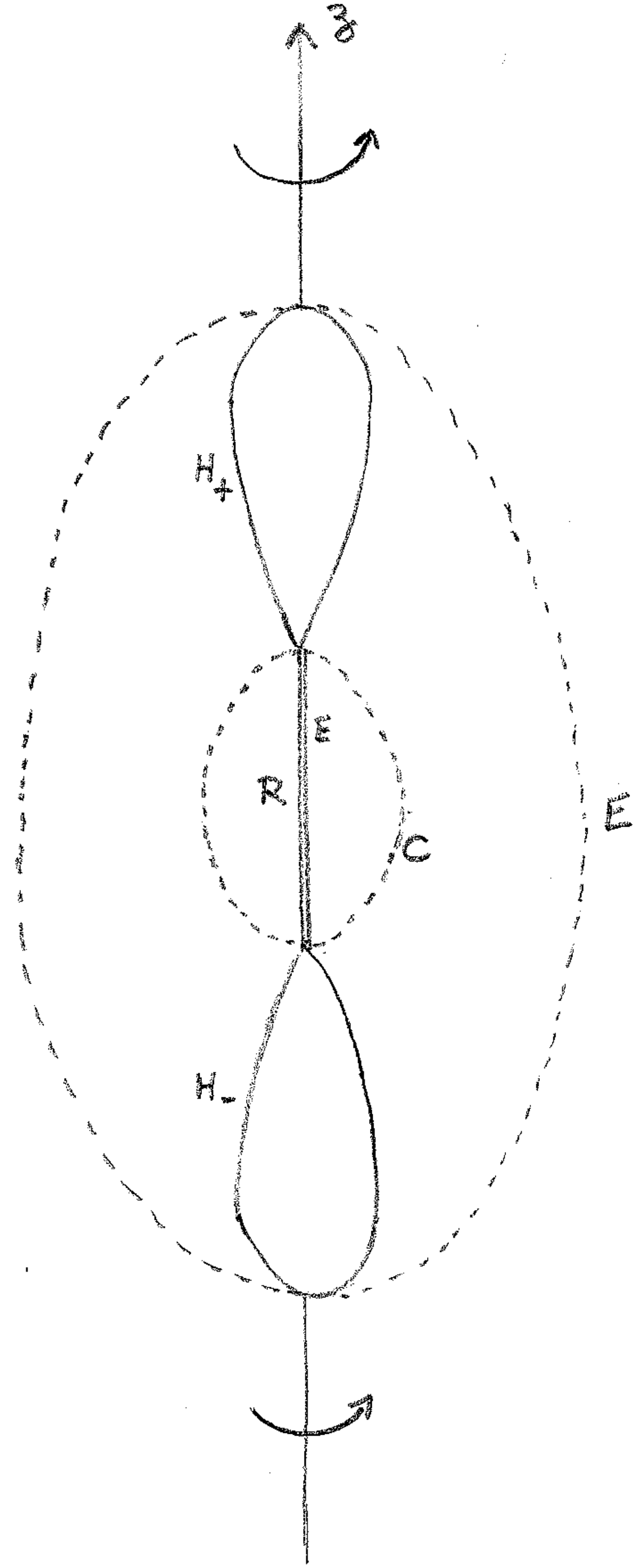
topologically S^2.

$l(0) = 1$, but $l(\pi) = \alpha \equiv \frac{2p\lambda(p)}{q^4} > \frac{8}{q^4}$

conical singularity!

– Horizon area:

$$\mathcal{A}_H = 4\pi \frac{\kappa^2 \lambda(p)}{2p} \simeq 4\pi M^2.$$



z
H+
E
R
C
E
H-

- The horizons: electromagnetic field

$$A_H = \varepsilon \left[\frac{q^2(2-p)}{2\lambda(p)} dt - \frac{\kappa q}{4} \frac{\delta(p) X^2 + q^2 \gamma(p) X^4}{\Sigma_H(X)} d\widehat{\varphi} \right]$$

– Near-horizon electric field, or

$$Q_H = -\frac{1}{4\pi} \oint_H \omega_H \, d\,\mathrm{Im}\psi \text{ (Tomimatsu1984)} \rightarrow$$

Electric charges $Q_+ = Q_- = -\dfrac{\varepsilon\kappa(1+p)}{2}$.

But the solution is electrically neutral,
so the rod must be also charged!

$\longrightarrow$ electric quadrupole

– Magnetic charges: $P_H = \dfrac{1}{4\pi} \oint_H dA_\varphi \rightarrow$

$$P_\pm = \pm \frac{\varepsilon\kappa\gamma(p)}{2q}.$$

$\longrightarrow$ magnetic dipole

• Komar mass and angular momentum:

$$M = \frac{1}{4\pi}\oint_\infty k^{\mu;\nu} d\Sigma_{\mu\nu} \quad (k = \partial_t)$$
$$J = -\frac{1}{8\pi}\oint_\infty l^{\mu;\nu} d\Sigma_{\mu\nu} \quad (l = \partial_\varphi)$$

– Ostrogradsky theorem $\longrightarrow$

$$M = \sum_n \frac{1}{4\pi}\oint_{H_n} k^{\mu;\nu} d\Sigma_{\mu\nu} + \frac{1}{4\pi}\int k^{\mu;\nu}{}_{;\nu} dS\mu$$
$$= \quad \cdots\cdots\cdots \quad -\frac{1}{4\pi}\int R^{\mu}{}_{\nu} k^{\nu}\, dS_\mu$$

– Tomimatsu: using the EM eqs., the bulk integral can be converted to a surface integral $\to$ total horizon mass $M_H = \frac{1}{4\pi}\oint_H \omega_H \, d\,\mathrm{Im}\mathcal{E}$:

$$M_+ = M_- = \frac{\kappa}{p} + \frac{\kappa p}{2}$$

$> M/2$, so the rod must have negative mass!

– Horizon angular momentum:

$$J_+ = J_- = -\frac{\kappa^2}{8qp}\left[2\lambda(2+p^2) - q^2 p(1+p)(2-p)\right]$$

– Also horizon NUT charges: $N_\pm = \pm\frac{\kappa\lambda(p)}{4q}$.

- The rod

Near-rod configuration ($\xi^2 \equiv x^2 - 1 \to 0$):

$$
\begin{aligned}
ds^2 &\sim -\frac{\kappa^2 q^2}{4}(1-y^2)^2\, d\varphi^2 + \frac{\kappa^2 q^4}{p^2(1-y^2)}\left[\frac{dy^2}{1-y^2}\right. \\
&+ \left. d\xi^2 + \alpha^2\Omega_H^2 \xi^2 \left(dt - \omega(y) d\varphi\right)^2\right], \\
A &\sim -\varepsilon\left[\left(1 - \frac{2(1+p)\xi^2}{q^2(1-y^2)}\right) dt + A_\varphi(y) d\varphi\right],
\end{aligned}
$$

$\to$ conical singularity, with finite Ricci square scalar

$$
R^{\mu\nu}R_{\mu\nu} \sim \frac{64 p^4}{\kappa^4 q^{12}}[(1+p)^2 + q^2 y^2]^2.
$$

– Transformation to the horizon co-rotating frame:

$$
\begin{aligned}
ds^2 &\sim q^4\left[-\frac{(1-y^2)^2}{4\lambda^2(p)}\left(dt + \Omega_H^{-1}\, d\hat{\varphi}\right)^2\right. \\
&+ \left. \frac{\kappa^2}{p^2(1-y^2)}\left(\frac{dy^2}{1-y^2} + d\xi^2 + \alpha^2 \xi^2\, d\hat{\varphi}^2\right)\right].
\end{aligned}
$$

• Interlude: Straight spinning cosmic string in flat spacetime:
(Deser, Jackiw, 't Hooft 1984, GC 1985)

$$d\hat{s}^2 = -(dt + 4J_S\, d\hat{\varphi})^2 + d\rho^2 + \alpha^2 \rho^2\, d\hat{\varphi}^2 + dz^2$$

($\alpha = 1 - 4M_S$).

The same viewed in a rotating frame,

$$d\varphi = d\hat{\varphi} - \Omega\, dt$$

with critical angular velocity $\Omega = -1/4J_S$:

$$ds^2 = \alpha^2\Omega^2\rho^2(dt + \Omega^{-1}\, d\varphi)^2 + d\rho^2 - \Omega^{-2} d\varphi^2 + dz^2.$$

– The rod is a spinning cosmic string in curved spacetime, with negative tension ($\alpha > 1$).

– This "spinning" string is a Misner string connecting 2 opposite NUT sources at $z = \pm\kappa$. We do not periodically identify time (Clément, Gal'tsov, Guenouche, "Rehabilitating space-times with NUTs", Phys. Lett. B750 (2015) 591, arXiv:1508.07622)

– 1 NUT source : $\omega = -2N\cos\theta$
2 opposite NUT sources
$\omega = -2N\cos\theta_+ + 2N\cos\theta_-$
On the axis ($\rho = 0$):
$\omega = -4N$ $(-\kappa < z < \kappa)$ or 0 $(|z| > \kappa)$ $\Rightarrow$

$$N = J_S = -\frac{1}{4\Omega_H} = \frac{\kappa\lambda(p)}{4q}$$

- Rod vector potential

$$A_\varphi \sim -\varepsilon\kappa \left[\frac{\gamma(p)}{q} + \frac{q(1-y^2)}{2}\right]$$

– Constant contribution $-\varepsilon\kappa\dfrac{\gamma(p)}{q} = P_- - P_+$:

The rod is also a Dirac string connecting
2 opposite magnetic monopoles at $z = \pm\kappa$

– Radial magnetic flux density

$$\sqrt{|g|}B^\xi = F_{y\varphi} = \varepsilon\kappa q y$$

$\longrightarrow$ rod magnetic moment

$$\mu_R = \frac{1}{4\pi}\int_{-1}^{+1} \sqrt{|g|}B^\xi \, z \, 2\pi \, dy = \frac{\varepsilon\kappa^2 q}{3} = \frac{\mu}{3}$$

- Rod electric charge

The Maxwell equation $\partial_\nu(\sqrt{|g|}F^{\mu\nu}) = 0$ is satisfied only outside sources $\rightarrow$ distributional contribution

$$Q_R = \frac{1}{4\pi}\int \left[\partial_\xi\left(\sqrt{|g|}F^{t\xi}\right)\right] d\xi\, dy\, d\varphi.$$

In the global frame,

$$A_t = -\varepsilon[1 + \mathrm{O}(\xi^2)], \quad g_{tt} = \mathrm{O}(\xi^2) \quad \rightarrow$$

$$F_{t\xi} \propto \xi \quad (\xi > 0), \quad \sqrt{|g|}F^{t\xi} \propto \theta(\xi)$$

$$\Rightarrow \quad Q_R = \frac{1}{4\pi}\int \varepsilon\kappa(1+p)\delta(\xi)\, d\xi\, dy\, d\varphi = \varepsilon\kappa(1+p)$$

ensures $Q_+ + Q_- + Q_R = 0$.

- Rod mass

The Einstein eqs. with Maxwell source $R_{\mu\nu} - 8\pi T_{\mu\nu} = 0$ are only satisfied outside sources. In the presence of distributional sources,

$$R_{\mu\nu} - 8\pi T_{\mu\nu} = [R_{\mu\nu}] - 8\pi[T_{\mu\nu}]$$

$\Rightarrow$ Komar mass at ∞

$$M = M_+ + M_- + M_R \quad \text{with}$$

$$M_R = -\frac{1}{4\pi}\int \left([R_t^t] - 8\pi[T_t^t]\right)\sqrt{|g|}\,d^3x = M_R^{\text{grav}} + M_R^{\text{em}}$$

$$[R_t^t] = -(g_{tt})^{-1/2} g^{ij} (g_{tt})^{1/2}_{,j;i} = -g^{\xi\xi}\xi^{-1}\delta(\xi)$$

$\rightarrow \quad M_R^{\text{grav}} = \kappa$

$$M_R^{\text{em}} = Q_R A_t(\xi = 0) = -\kappa(1 + p)$$

$-$Total rod mass $M_R = -\kappa p$
repulses test particles (antigravity)

$$\frac{2\kappa}{p} = 2\left(\frac{\kappa}{p} + \frac{\kappa p}{2}\right) - \kappa p$$

• Behind the horizon H_+:
– Region II_+ with $-1 < x < 1$ and $y > 1$
– Inner horizon H'_+ $(x = -1,\ y = 1)$

• Between outer and inner horizon,
timelike singularity S_0 ($y = \infty$), with $f < 0$,
$g_{\varphi\varphi} < 0$, and Ricci square scalar $\sim y^4$

• Is S_0 really at infinity?
– The 2 horizons H and H' are topological spheres, with $\mathcal{A} > \mathcal{A}'$
– Near S_0, putting $y = \eta^{-1}$, $x = \cos\chi$,

$$\begin{aligned} ds^2 &\simeq -\frac{a}{\cos^2\chi}\eta^{-4}\,d\varphi^2 + b\cos^2\chi[d\eta^2 + \eta^2\,d\chi^2 \\ &+ c\eta^2\sin^2\chi(dt + k(x)\eta^{-2}d\varphi^2)] \end{aligned}$$

$\Longrightarrow \sqrt{|g|}$ goes to a finite limit for $y \to \infty$
– $\eta = 0$ is a "point" (timelike line)

• Only spacelike geos with $E = 0$ terminate at $y = \infty$.

• Now the z axis ($\rho = 0$) includes:
– a regular segment $y = 1$, $-1 < x < 1$ between the 2 horizons;
– 2 singular rods $x = \pm 1$ from the outer or inner horizon to S_0.
– The 2 rods have different tensions and angular velocities.
– The 2 rods carry different diverging electric charges (integration on y from 1 to ∞), so S_0 must carry infinite electric charge, and finite magnetic and NUT charges.

• Similar region II_- ($y < -1$) behind H_-.

• Beyond the inner horizons:
– Region III ($X < -1$, $-1 \leq y \leq 1$), with a singular rod connecting the 2 co-rotating horizons H'_+ and H'_-, and
– A timelike ring singularity $\Sigma(x_0, 0) = 0$, with $f > 0$, $g_{\varphi\varphi} < 0$.
– Only fine-tuned spacelike geodesics can reach this ring (similar to Kerr-Newman).

• Summary

– Exact one-parameter rotating e.m. solution generated from ZV2 static vacuum solution.
– No naked ring singularity: more regular than the seed static solution, or its rotating vacuum counterpart TS2.
– Has only dipole magnetic moment and quadrupole electric moment.
– Generated by a complex system: 2 co-rotating dyonic NUTty black holes held apart by a rotating, electrically charged, magnetized rod.
– More general 4-parameter class of solutions (Manko et al2000): Does it include a purely magnetic rotating solution (without quadrupole electric moment) $\longrightarrow$ a system of 2 magnetic NUTty black holes and an electrically neutral rod?