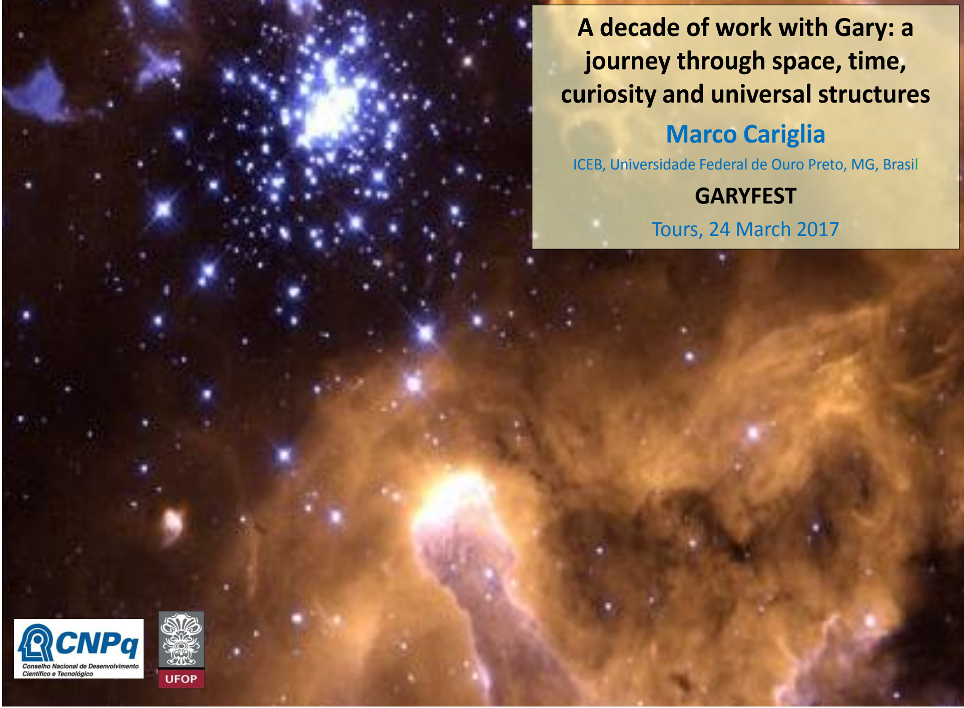
A decade of work with Gary: a journey through space, time, curiosity and universal structures

Marco Cariglia

ICEB, Universidade Federal de Ouro Preto, MG, Brasil

GARYFEST

Tours, 24 March 2017

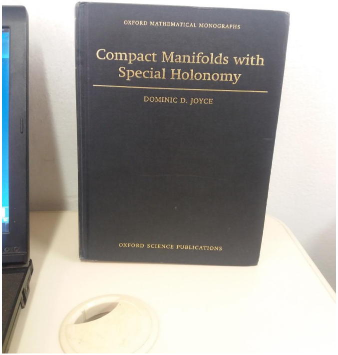


CNPq Conselho Nacional de Desenvolvimento Científico e Tecnológico

UFOP

2001-2002

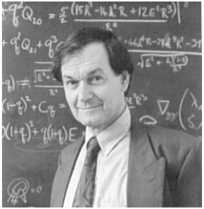

$T_H = \frac{hc^3}{8\pi GMk}$
and black holes gave forth light!


OXFORD MATHEMATICAL MONOGRAPHS
Compact Manifolds with Special Holonomy
DOMINIC D. JOYCE
OXFORD SCIENCE PUBLICATIONS

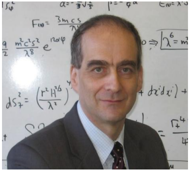
Summary

Many contributors



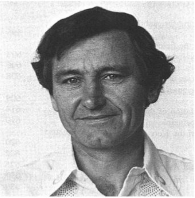


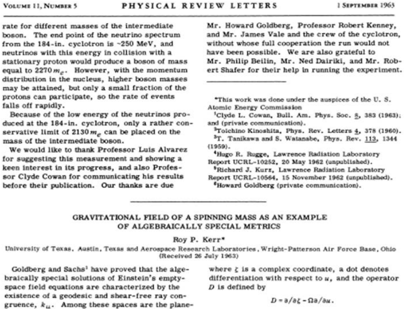




Kerr's solution

In 1963 Roy Kerr discovers the solution of Einstein equations describing a stationary rotating black hole.





VOLUME 11, NUMBER 5 PHYSICAL REVIEW LETTERS 1 SEPTEMBER 1963

rate for different masses of the intermediate boson. The end point of the neutrino spectrum from the 184-in. cyclotron is ~250 MeV, and neutrinos with this energy in collision with a stationary proton would produce a boson of mass equal to 2270 m_e. However, with the momentum distribution in the nucleus, higher boson masses may be attained, but only a small fraction of the protons can participate, so the rate of events falls off rapidly.

Because of the low energy of the neutrinos produced at the 184-in. cyclotron, only a rather conservative limit of 2130 m_e can be placed on the mass of the intermediate boson.

We would like to thank Professor Luis Alvarez for suggesting this measurement and showing a keen interest in its progress, and also Professor Clyde Cowan for communicating his results before their publication. Our thanks are due Mr. Howard Goldberg, Professor Robert Kenney, and Mr. James Vale and the crew of the cyclotron, without whose full cooperation the run would not have been possible. We are also grateful to Mr. Philip Beilin, Mr. Ned Dairiki, and Mr. Robert Shafer for their help in running the experiment.

*This work was done under the auspices of the U. S. Atomic Energy Commission

[1]Clyde L. Cowan, Bull. Am. Phys. Soc. 8, 383 (1963); and (private communication).

[2]Toichino Kinoshita, Phys. Rev. Letters 4, 378 (1960).

[3]T. Tanikawa and S. Watanabe, Phys. Rev. 113, 1344 (1959).

[4]Hugo R. Rugge, Lawrence Radiation Laboratory Report UCRL-10252, 20 May 1962 (unpublished).

[5]Richard J. Kurz, Lawrence Radiation Laboratory Report UCRL-10564, 15 November 1962 (unpublished).

[6]Howard Goldberg (private communication).

GRAVITATIONAL FIELD OF A SPINNING MASS AS AN EXAMPLE OF ALGEBRAICALLY SPECIAL METRICS

Roy P. Kerr*
University of Texas, Austin, Texas and Aerospace Research Laboratories, Wright-Patterson Air Force Base, Ohio
(Received 26 July 1963)

Goldberg and Sachs[1] have proved that the algebraically special solutions of Einstein's empty-space field equations are characterized by the existence of a geodesic and shear-free ray congruence, k_μ. Among these spaces are the plane-

where ζ is a complex coordinate, a dot denotes differentiation with respect to u, and the operator D is defined by

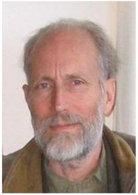
$$D = \partial/\partial\zeta - \Omega\partial/\partial u.$$

$$ds^2 = -\frac{\Delta}{\Sigma}\left(dt - a\sin^2\theta d\varphi\right)^2 + \frac{\sin^2\theta}{\Sigma}\left[(r^2+a^2)d\varphi - a dt\right]^2 + \frac{\Sigma}{\Delta}dr^2 + \Sigma\, d\theta^2\,,$$

where $\Sigma = r^2 + a^2\cos^2\theta$, $\Delta = r^2 - 2Mr + a^2$.

Carter’s constant

In 1968 Brandon Carter obtains the ’unexpected separation’ of the Hamilton-Jacobi equations in Kerr-Newman and shows the existence of a fourth constant of motion



PHYSICAL REVIEW VOLUME 174, NUMBER 5 25 OCTOBER 1968

Global Structure of the Kerr Family of Gravitational Fields

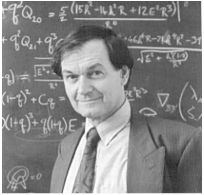
BRANDON CARTER*
Department of Applied Mathematics and Theoretical Physics, University of Cambridge, Cambridge, England

(Received 29 March 1968)

The Kerr family of solutions of the Einstein and Einstein-Maxwell equations is the most general class of solutions known at present which could represent the field of a rotating neutral or electrically charged body in asymptotically flat space. When the charge and specific angular momentum are small compared with the mass, the part of the manifold which is stationary in the strict sense is incomplete at a Killing horizon. Analytically extended manifolds are constructed in order to remove this incompleteness. Some general methods for the analysis of causal behavior are described and applied. It is shown that in all except the spherically symmetric cases there is nontrivial causality violation, i.e., there are closed timelike lines which are not removable by taking a covering space; moreover, when the charge or angular momentum is so large that there are no Killing horizons, this causality violation is of the most flagrant possible kind in that it is possible to connect any event to any other by a future-directed timelike line. Although the symmetries provide only three constants of the motion, a fourth one turns out to be obtainable from the unexpected separability of the Hamilton-Jacobi equation, with the result that the equations, not only of geodesics but also of charged-particle orbits, can be integrated completely in terms of explicit quadratures. This makes it possible to prove that in the extended manifolds all geodesics which do not reach the central ring singularities are complete, and also that those timelike or null geodesics which do reach the singularities are entirely confined to the equator, with the further restriction, in the charged case, that they be null with a certain uniquely determined direction. The physical significance of these results is briefly discussed.

Killing tensor

In 1970 Walker and Penrose show that Carter's constant arises from a symmetric *Killing-Stäckel* tensor $\nabla_{(\lambda}K_{\mu\nu)} = 0$:



Commun. math. Phys. 18, 265—274 (1970)

On Quadratic First Integrals of the Geodesic Equations for Type {22} Spacetimes

MARTIN WALKER and ROGER PENROSE
Department of Mathematics, Birkbeck College, London, England

Received May 1, 1970

Abstract. It is shown that every type {22} vacuum solution of Einstein's equations admits a quadratic first integral of the null geodesic equations (conformal Killing tensor of valence 2), which is independent of the metric and of any Killing vectors arising from symmetries. In particular, the charged Kerr solution (with or without cosmological constant) is shown to admit a Killing tensor of valence 2. The Killing tensor, together with the metric and the two Killing vectors, provides a method of explicitly integrating the geodesics of the (charged) Kerr solution, thus shedding some light on a result due to Carter.

$$C = \frac{1}{2} K^{\mu\nu} p_\mu p_\nu \,,$$

commutes with Hamiltonian $H = \frac{1}{2} g^{\mu\nu} p_\mu p_\nu$.

Floyd’s result

In his Ph.D. thesis "The Dynamics of Kerr Fields" in 1973 Floyd shows that the Killing tensor in Kerr admits a ’square root’ in terms of a 2–form:

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$$K_{\mu\nu} = f_{\mu\rho} f_{\nu}{}^{\rho} \, .$$

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$$K_{\mu\nu} = f_{\mu\rho} f_{\nu}{}^{\rho} \, .$$

$f_{\mu\nu} = -f_{\nu\mu}$ satisfies

$$\nabla_{(\lambda} f_{\mu)\nu} = 0 \leftrightarrow \nabla_{\lambda} f_{\mu\nu} = \nabla_{[\lambda} f_{\mu\nu]} \, ,$$

and today is called an antisymmetric *Killing-Yano* tensor.

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Classically the vector $f^{\mu\nu} p_{\nu}$ is parallely propagated along geodesics and squares to $K^{\mu\nu} p_{\mu} p_{\nu}$.

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Existence of the Killing-Yano tensor implies that spacetime must be of type D [Collinson 1974].

Yano's tensor

What we now call a Killing-Yano tensor was introduced by Yano in 1951 to generalise Killing vector fields and some results on their existence.

ANNALS OF MATHEMATICS
Vol. 55, No. 2, March, 1952
Printed in U.S.A.

SOME REMARKS ON TENSOR FIELDS AND CURVATURE

BY KENTARO YANO

(Received April 5, 1951)

§1. Introduction

S. Bochner [1, 2, 3] has recently developed a beautiful theory on curvature and Betti numbers of an orientable compact Riemannian space V_n with positive definite metric. He starts from the lemma:

Separation of variables for Klein-Gordon

Again Carter in 1977 separates variables for the Klein-Gordon equation using an operator built from the Killing-Stäckel tensor:

PHYSICAL REVIEW D VOLUME 16, NUMBER 12 15 DECEMBER 1977

Killing tensor quantum numbers and conserved currents in curved space

Brandon Carter

CNRS E. R. 176, Départment d'Astrophysique Fondamentale, Observatoire de Paris, 92190 Meudon, France

(Received 7 April 1977)

The relationship between relativistic quantum current conservation laws in a curved-space background and the corresponding "good quantum numbers," i.e., operators that commute with the fundamental wave operator in a first-quantized field theory, is considered. It is shown that under favorable circumstances (such as vanishing Ricci curvature) the existence of such an operator for scalar fields is automatically implied by the existence of the corresponding constant for particle trajectories in the classical limit, that is to say, by the existence of a Killing vector or a "Killing tensor" in the first- and second-order cases, respectively. Thus the fourth constant of the motion for a scalar quantum field in the Kerr metric background arises automatically from the Killing tensor defining the fourth constant of the classical motion. Another application is to the Runge-Lenz constants in the nonrelativistic hydrogen atom problem. The "Schiff conjecture" concerning the relationship between classical mechanics and first-quantized field theory in connection with the equivalence principle is discussed in passing.

$$\hat{K} = D_\mu K^{\mu\nu} D_\nu \,,$$

$$R_{\mu\nu} = 0 \rightarrow \left[\hat{K}, D_\mu g^{\mu\nu} D_\nu\right] = 0 \,.$$

Benenti's theory of separation of variables

ACCADEMIA NAZIONALE DEI LINCEI

Estratto dai *Rendiconti della Classe di Scienze fisiche, matematiche e naturali*

Serie VIII, vol. LXII, fasc. 1 - Gennaio 1977

Fisica matematica. — *Integrabilità per separazione delle variabili delle equazioni alle derivate parziali lineari del secondo ordine interessanti la fisica–matematica* (*). Nota di Sergio Benenti, presentata (**) dal Socio C. Agostinelli.

Summary. — We state a local characterization of the Riemannian manifolds upon which second order linear partial differential equations of mathematical physics are integrable by separation of the variables. Among the results we have a generalizazion of a classical theorem of Eisenhart on the separability of the Schrödinger equation in orthogonal coordinates.

Benenti in the 70s-80s, and later with collaborators, set out a theory for the separation of Hamilton-Jacobi equation, Schrödinger and Klein-Gordon. Also Kalnins and Miller, Francaviglia.

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Main ingredients are *Killing vectors* and rank-2 *Killing-Stäckel tensors*, mutually compatible.

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Main ingredients are *Killing vectors* and rank-2 *Killing-Stäckel tensors*, mutually compatible.

If spacetime is curved compatibility with Ricci tensor necessary. If a potential V is present then this must also be compatible.

Total agular momentum operator for the Dirac equation

Found by Carter and McLenaghan in 1979

PHYSICAL REVIEW D VOLUME 19, NUMBER 4 15 FEBRUARY 1979

Generalized total angular momentum operator for the Dirac equation in curved space-time

B. Carter

Groupe d'Astrophysique Relativiste, Observatoire de Paris, Meudon, France

R. G. McLenaghan

Departement de Mathematique, Universite Libre de Bruxelles and Department of Applied Mathematics, University of Waterloo, Ontario, Canada

(Received 7 August 1978)

It is found that an operator of the form $i\gamma_5\gamma^\mu[f_\mu{}^\nu\nabla_\nu-(1/6)\gamma^\nu\gamma^\rho f_{\mu\nu;\rho}]$ commutes with the Dirac operator $\gamma^\mu\nabla_\nu$ whenever $f_{\mu\nu}$ is an antisymmetric tensor satisfying the Penrose-Floyd equation $f_{\mu(\nu;\rho)}=0$. Such a tensor exists notably in the Kerr solutions and in the flat-space limit wherein the operator can be interpreted as the square root of the ordinary total squared angular momentum Casimir operator of the rotation group.

Built using Floyd's tensor.

Similar results for the spinning particle in the ’90s

Gibbons, Rietdijk and van Holten show that the Killing-Yano tensor in Kerr also generates symmetries in the theory of the spinning particle.

Nuclear Physics B404 (1993) 42–64
North-Holland

NUCLEAR PHYSICS B

SUSY in the sky

G.W. Gibbons [1]
DAMTP, Cambridge, UK

R.H. Rietdijk [2]
Physics Department, University of Durham, Durham, UK

J.W. van Holten [3]
NIKHEF-H, NL-1009 DB Amsterdam, The Netherlands

Received 23 March 1993
Accepted for publication 6 April 1993

Spinning particles in curved space–time can have fermionic symmetries generated by the square root of bosonic constants of motion other than the hamiltonian. We present a general analysis of the conditions under which such new supersymmetries appear, and discuss the Poisson–Dirac algebra of the resulting set of charges, including the conditions of closure of the new algebra. An example of a new non-trivial supersymmetry is found in black-hole solutions of the Kerr–Newman type and corresponds to the Killing–Yano tensor, which plays an important role in solving the Dirac equation in these black-hole metrics.

Quantum Mechanical Result

Killing-Yano tensors of generic rank generate symmetries for the Dirac equation

INSTITUTE OF PHYSICS PUBLISHING CLASSICAL AND QUANTUM GRAVITY

Class. Quantum Grav. **21** (2004) 1051–1077 PII: S0264-9381(04)64827-1

Quantum mechanics of Yano tensors: Dirac equation in curved spacetime

Marco Cariglia

DAMTP, Centre for Mathematical Sciences, Cambridge University, Wilberforce Road, Cambridge CB3 OWA, UK

E-mail: M.Cariglia@damtp.cam.ac.uk

Received 17 June 2003
Published 22 January 2004

INSTITUTE OF PHYSICS PUBLISHING CLASSICAL AND QUANTUM GRAVITY

Class. Quantum Grav. 21 (2004) 3171–3195 PII: S0264-9381(04)76774-X

The general form of supersymmetric solutions of $N = (1, 0)$ $U(1)$ and $SU(2)$ gauged supergravities in six dimensions

Marco Cariglia and Oisín A P Mac Conamhna

DAMTP, Centre for Mathematical Sciences, University of Cambridge, Wilberforce Road, Cambridge CB3 0WA, UK

E-mail: M.Cariglia@damtp.cam.ac.uk and O.A.P.MacConamhna@damtp.cam.ac.uk

5. Fluxes and intrinsic torsion

$$\delta\psi_\mu^A = \nabla_\mu^T \epsilon^A = \left[\partial_\mu + \tfrac{1}{4}(\omega_{\mu\hat{\alpha}\hat{\beta}} + T_{\mu\hat{\alpha}\hat{\beta}})\Gamma^{\hat{\alpha}\hat{\beta}}\right]\epsilon^A,$$

$$\begin{aligned}
\mathcal{W}_2 &= \tfrac{1}{4}(\Omega \lrcorner \mathrm{d}\Omega + \overline{\Omega} \lrcorner \mathrm{d}\overline{\Omega}),\\
\mathcal{W}_4 &= J \lrcorner \mathrm{d}J,\\
\mathcal{W}_5 &= \tfrac{1}{4}(\Omega \lrcorner \mathrm{d}\overline{\Omega} + \overline{\Omega} \lrcorner \mathrm{d}\Omega),
\end{aligned}$$

$$\begin{aligned}
\mathcal{W}_2 &= \tfrac{g}{2}(A^2J^2 - A^3J^3)\\
\mathcal{W}_4 &= -g(A^2J^2 + A^3J^3) + H\partial_u(H^{-1}\beta),\\
\mathcal{W}_5 &= -\tfrac{g}{2}(A^2J^2 + A^3J^3) - gA^1J^1 + H\partial_u(H^{-1}\beta).
\end{aligned}$$


Out to
Lunch
BACK
SOON


CITY OF
LONDON


UBS
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UBS
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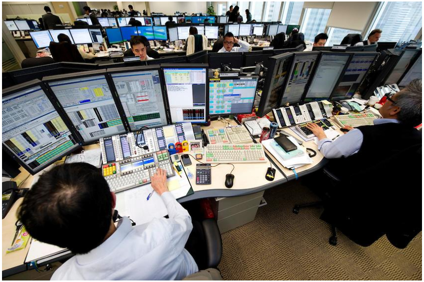
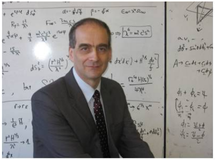


The Kerr-NUT-(A)dS solution

$n \geq 4$-dimensional, rotating, with cosmological constant [Myers, Perry 1986; Gibbons, Lü, Page, Pope 2004-2005; Chen, Lü, Pope 2006]



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Class. Quantum Grav. **23** (2006) 5323–5340 doi:10.1088/0264-9381/23/17/013

General Kerr–NUT–AdS metrics in all dimensions

W Chen, H Lü and C N Pope

George P & Cynthia W Mitchell Institute for Fundamental Physics, Texas A&M University, College Station, TX 77843-4242, USA

Received 24 May 2006, in final form 11 July 2006
Published 10 August 2006

Miracolous properties of Kerr persist in higher dimension


In 2007 Valeri Frolov and David Kubizňák find a Killing-Yano tensor in Kerr-NUT-(A)dS.

- Tower of $[n/2]$ Killing tensors and $[(n+1)/2]$ Killing vectors. Hamilton-Jacobi and Schrödinger equations separate [Frolov, Krtouš, Kubizňák 2007; Frolov, Krtouš, Kubizňák, Page 2007; Krtouš, Kubizňák, Page, Vasudevan 2007]

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- Tower of Killing-Yano tensors [Yano 1952; Gibbons, Rietdijk, van Holten 1992; Frolov, Kubizňák 2007]. The Dirac equation separates, the spinning particle is integrable [Oota Yasui 2008; Cariglia, Krtouš, Kubizňák 2011; Cariglia, Kubizňák 2012]

Miracolous properties of Kerr persist in higher dimension


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- Generalisation to Killing-Yano tensors with fluxes [Kubizňák, Kunduri, Yasui 2009; Houri, Kubizňák, Warnick, Yasui 2010-2012]

More on Kerr-NUT-(A)dS

PHYSICAL REVIEW D **76**, 084034 (2007)

Constants of geodesic motion in higher-dimensional black-hole spacetimes

Pavel Krtouš,[1,*] David Kubizňák,[2,1,†] Don N. Page,[2,‡] and Muraari Vasudevan[2,3,§]
[1]*Institute of Theoretical Physics, Charles University, V Holešovičkách 2, Prague, Czech Republic*
[2]*Theoretical Physics Institute, University of Alberta, Edmonton, Alberta, Canada T6G 2G7*
[3]*JLR Engineering, 111 SE Everett Mall Way, E-201, Everett, Washington 98208-3236, USA*
(Received 2 July 2007; published 25 October 2007)

In [Phys. Rev. Lett. **98**, 061102 (2007)], we announced the complete integrability of geodesic motion in the general higher-dimensional rotating black-hole spacetimes. In the present paper we prove all the necessary steps leading to this conclusion. In particular, we demonstrate the independence of the constants of motion and the fact that they Poisson commute. The relation to a different set of constants of motion constructed in [J. High Energy Phys. 02 (2007) 004] is also briefly discussed.

Higher dimensional Dirac

PHYSICAL REVIEW D **84**, 024008 (2011)

Dirac equation in Kerr-NUT-(A)dS spacetimes: Intrinsic characterization of separability in all dimensions

Marco Cariglia*
Departamento de Física, ICEB, Universidade Federal de Ouro Preto, Campus Morro do Cruzeiro, Morro do Cruzeiro, 35400-000 - Ouro Preto, MG - Brasil

Pavel Krtouš†
Institute of Theoretical Physics, Faculty of Mathematics and Physics, Charles University in Prague, V Holešovičkách 2, Prague, Czech Republic

David Kubizňák‡
DAMTP, University of Cambridge, Wilberforce Road, Cambridge CB3 0WA, United Kingdom
(Received 28 April 2011; published 6 July 2011)

We intrinsically characterize separability of the Dirac equation in Kerr-NUT-(A)dS spacetimes in all dimensions. Namely, we explicitly demonstrate that, in such spacetimes, there exists a complete set of first-order mutually commuting operators, one of which is the Dirac operator, that allows for common eigenfunctions which can be found in a separated form and correspond precisely to the general solution of the Dirac equation found by Oota and Yasui [Phys. Lett. B **659**, 688 (2008)]. Since all the operators in the set can be generated from the principal conformal Killing-Yano tensor, this establishes the (up-to-now) missing link among the existence of hidden symmetry, presence of a complete set of commuting operators, and separability of the Dirac equation in these spacetimes.

DOI: 10.1103/PhysRevD.84.024008 PACS numbers: 04.50.−h, 04.20.Jb, 04.50.Gh, 04.70.Bw

Separability for Dirac in Kerr-NUT-(A)dS

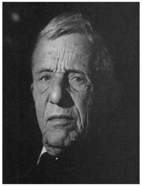
Summary

Eisenhart's result

- Luther Pfahler Eisenhart (1876-1965)

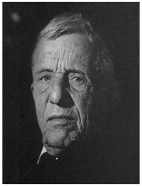
Eisenhart's result

- Luther Pfahler Eisenhart (1876-1965)



Eisenhart's result

- Luther Pfahler Eisenhart (1876-1965)

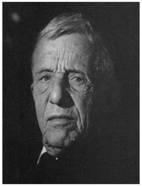


- Hamiltonian system with scalar potential $\Phi(q,t)$, and vector potential $A_i(q,t)$

$$H = \frac{1}{2} h^{ij}(q)\,(p_i + eA_i)\,(p_j + eA_j) + e^2\Phi\,.$$

Eisenhart's result

- Luther Pfahler Eisenhart (1876-1965)



- Hamiltonian system with scalar potential $\Phi(q,t)$, and vector potential $A_i(q,t)$

$$H = \frac{1}{2} h^{ij}(q)\,(p_i + eA_i)\,(p_j + eA_j) + e^2\Phi\,.$$

- Solutions $t \mapsto q^i(t)$, $i = 1,\ldots,d$, in 1 to 1 correspondence with *null geodesics* of Lorentzian metric in $(D+2)$-dimensions

$$ds^2 = \hat{g}_{AB}\,dy^A dy^B = h_{ij}\,dq^i dq^j + 2du\left(dv - \Phi du + A_i\,dq^i\right)\,.$$

Eisenhart's result

Study dynamics using Lorentzian geometry and, viceversa, obtain geometrical results from known dynamical systems.

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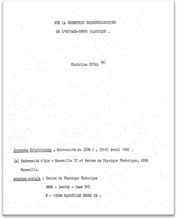
Example: Ricc-flat spaces in $(2, 2)$ signature with higher order Killing tensors using Drach systems [Cariglia and Galajinsky 2015].

Modern point of view



SUR LA GEOMETRIE CHRONOPROJECTIVE
DE L'ESPACE-TEMPS CLASSIQUE .

Christian DUVAL (*)

Journées Relativistes , Université de LYON 1 , 23-25 Avril 1982 .

(*) Université d'Aix - Marseille II et Centre de Physique Théorique, CNRS Marseille.

Adresse postale : Centre de Physique Théorique
CNRS - Luminy - Case 907.
F - 13288 MARSEILLE CEDEX 09 .

- Christian Duval indipendently discovers the lift and applies it to non-relativistic mechanics

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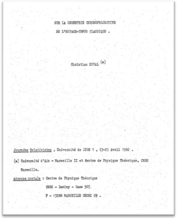

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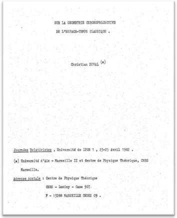

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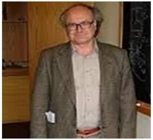
- More work done: Schrödinger equation, non-relativistic electrodynamics [Duval, Gibbons, Horváthy 1991], pp-waves, non-relativistic holography [Balasubramanian, McGreevy 2008; Son 2008; Duval, Hassaïne, Horváthy 2009; Bekaert, Morand 2013], Lorentzian distance and minimal action [Minguzzi 2007], . . .

Bargmann structure and conformal symmetries



PHYSICAL REVIEW D VOLUME 43, NUMBER 12 15 JUNE 1991

Celestial mechanics, conformal structures, and gravitational waves

Christian Duval
Centre de Physique Théorique, Centre National de la Recerché Scientifique, Luminy, and Université d'Aix-Marseille II, case 907, F-13288 Marseille, CEDEX 9, France

Gary Gibbons
Department of Applied Mathematics and Theoretical Physics, Silver Street, Cambridge CB3 9EW, United Kingdom

Péter Horváthy
Département de Mathématiques, Faculté des Sciences, Parc de Grandmont, F-37200 Tours, France

(Received 27 November 1990)

Newton's equations for the motion of N nonrelativistic point particles attracting according to the inverse square law may be cast in the form of equations for null geodesics in a $(3N + 2)$-dimensional Lorentzian spacetime which is Ricci flat and admits a covariantly constant null vector. Such a spacetime admits a Bargmann structure and corresponds physically to a plane-fronted gravitational wave (generalized pp wave). Bargmann electromagnetism in five dimensions actually comprises the two distinct Galilean electromagnetic theories pointed out by Le Bellac and Lévy-Leblond. At the quantum level, the N-body Schrödinger equation may be cast into the form of a massless wave equation. We exploit the conformal symmetries of such spacetimes to discuss some properties of the Newtonian N-body problem, in particular, (i) homographic solutions, (ii) the virial theorem, (iii) Kepler's third law, (iv) the Lagrange-Laplace-Runge-Lenz vector arising from three conformal Killing two-tensors, and (v) the motion

Modern point of view 2

- Relation with dynamics best seen using the inverse metric .

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Exemples: Kepler problem with $G(t)$, quantum dots, spinning tops, susy, curved space [van Holten 2007; Horváthy, Ngome 2009; Ngome 2009; van Holten, Horváthy, Ngome 2010; Visinescu 2010 & 2011; Igata, Koike, Ishihara 2011; Gibbons, Houri, Kubizňák, Warnick 2011; Cariglia, Kubizňák 2012; Cariglia, Gibbons, van Holten, Horváthy, Zhang 2014]

Application to non-relativistic holography

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The geometry of Schrödinger symmetry in non-relativistic CFT

C. Duval [a,1], M. Hassaïne [b], P.A. Horváthy [c,*]

[a] *Centre de Physique Théorique, CNRS, Luminy, Case 907, F-13288 Marseille Cedex 9, France*
[b] *Instituto de Matematica y Física, Universidad de Talca, Casilla 747, Talca, Chile*
[c] *Laboratoire de Mathématiques et de Physique, Théorique, Université de Tours, Parc de Grandmont, F-37200 Tours, France*

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ABSTRACT

The non-relativistic conformal "Schrödinger" symmetry of some gravity backgrounds proposed recently in the AdS/CFT context, is explained in the "Bargmann framework". The formalism incorporates the Equivalence Principle. Newton–Hooke conformal symmetries, which are analogs of those of Schrödinger in the presence of a negative cosmological constant, are discussed in a similar way.

Summary

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In fact there are several ways to have a null lift.

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with $\hat{p}_T^2 = |E|$.

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Example 3: Eisenhart lift of Toda chain [Cariglia, Gibbons 2014]

Projective nature of null dynamics [Cariglia AOP 2015]

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Hamiltonian H of degree 2 in momenta	$\rightarrow$	$\mathcal{H} = \frac{1}{2}\hat{g}^{AB}\hat{p}_A\hat{p}_B$ null lift: non-degenerate, indefinite, quadratic, homogeneous

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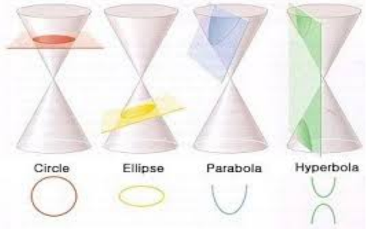

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Unifying perspective: coupling-constant metamorphosis, temporal reparameterisation, Schrödinger transformations are *all projetive*.

CCM

$$H = H_0(q,p) - gF(q)$$

$$G = \frac{H_0}{F} - \frac{h}{F}$$

PHYSICAL REVIEW

LETTERS

VOLUME 53 29 OCTOBER 1984 NUMBER 18

Coupling-Constant Metamorphosis and Duality between Integrable Hamiltonian Systems

J. Hietarinta
Wihuri Physical Laboratory and Department of Physical Sciences, University of Turku, 20500 Turku, Finland

and

B. Grammaticos and B. Dorizzi
Département des Mathématiques (MTI), Centre National d'Études de Télécommunications, F-92131 Issy-les-Moulineaux, France

and

A. Ramani
Centre de Physique Théorique, École Polytechnique, F-91128 Palaiseau, France

(Received 3 August 1984)

We introduce a noncanonical ("new-time") transformation which exchanges the roles of a coupling constant and the energy in Hamiltonian systems while preserving integrability. In this way we can construct new integrable systems and, for example, explain the observed duality between the Hénon-Heiles and Holt models. It is shown that the transformation can sometimes connect weak- and full-Painlevé Hamiltonians. We also discuss quantum integrability and find the origin of the deformation $-\frac{5}{72}\hbar^2 x^{-2}$.

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Example n.2: Kepler problem projectively equivalent to null geodesic motion in conformally-flat Minkowski space [Cariglia JGP 2016].

Summary

1. Hidden Symmetries and Special Tensors
2. The Eisenhart-Duval lift
3. Projective Mechanics
4. Non-relativistic fermions in condensed matter

Recent work

arXiv:1611.06254
Curvatronics with bilayer graphene in an effective $4D$ spacetime

Collaborators

Andrea Perali,
Camerino

Roberto Giambò,
Camerino

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New answer: electrons at the Dirac points feel the intrinsic curvature of the surface. Modulating this allows opening and closing a gap: key effect of curvatronics.

Tight binding model

In matrix form

$$\begin{pmatrix} E & -tf(\vec{q}) \\ -tf^*(\vec{q}) & E \end{pmatrix} \begin{pmatrix} \psi_A \\ \psi_B \end{pmatrix} = 0\,,$$

$$f(\vec{q}) = \sum_{j=1}^{3} e^{i\,\vec{q}\cdot\vec{\delta}_j}\,.$$

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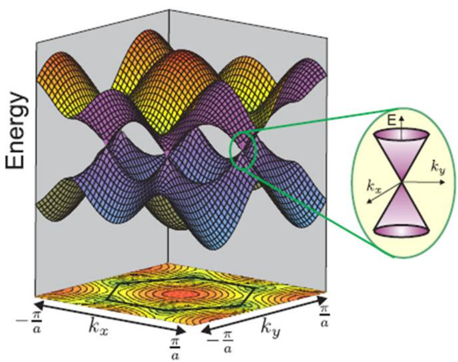
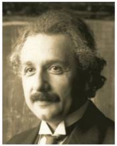


Here live Dirac quasiparticles



Continuum limit: Dirac equation

Dirac points

$$\vec{K} = \frac{2\pi}{3a}\left(1, \frac{1}{\sqrt{3}}\right) , \qquad \vec{K}' = \frac{2\pi}{3a}\left(1, -\frac{1}{\sqrt{3}}\right) .$$

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$$H_{eff} = \begin{pmatrix} 0 & tf(\vec{k}) \\ tf^*(\vec{k}) & 0 \end{pmatrix},$$

where $\vec{q} = \vec{k} + \vec{K}$, and $f(\vec{k}) = \frac{3}{2}ae^{i\frac{\pi}{6}}(k_x - ik_y)$.

Continuum limit: Dirac equation

Dirac points

$$\vec{K} = \frac{2\pi}{3a}\left(1, \frac{1}{\sqrt{3}}\right) , \qquad \vec{K}' = \frac{2\pi}{3a}\left(1, -\frac{1}{\sqrt{3}}\right) .$$

$a = 0.142$nm is carbon-carbon distance.

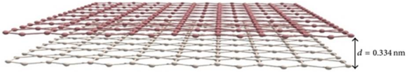
$$H_{eff} = \begin{pmatrix} 0 & tf(\vec{k}) \\ tf^*(\vec{k}) & 0 \end{pmatrix} ,$$

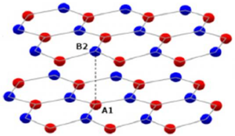
where $\vec{q} = \vec{k} + \vec{K}$, and $f(\vec{k}) = \frac{3}{2} a e^{i\frac{\pi}{6}} (k_x - ik_y)$. Equivalent form

$$\begin{aligned} H^K_{eff} &= \hbar v_F (\sigma_x k_x + \sigma_y k_y) , \\ H^{K'}_{eff} &= \hbar v_F (-\sigma_x k_x + \sigma_y k_y) , \end{aligned}$$

$v_F = \frac{3at}{2\hbar} \sim 10^6 ms^{-1}$ is the Fermi velocity.

AB stacked bilayer graphene





$\gamma = 0.39\text{eV}$

Tight-binding model

Low energy limit tight-binding model Hamiltonian of AB stacked bilayer graphene close to the K or K' points

$$H_{K,K'} = \begin{pmatrix} 0 & \hbar v_F \kappa & 0 & \gamma \\ \hbar v_F \bar{\kappa} & 0 & 0 & 0 \\ 0 & 0 & 0 & \hbar v_F \kappa \\ \gamma & 0 & \hbar v_F \bar{\kappa} & 0 \end{pmatrix}$$

$\kappa = \tau k_x + i k_y$ wave number of the excitation.
$\tau = \pm 1$ denotes Hamiltonian relative to the K or K' point
$\gamma \sim 0.39 eV$ hopping parameter between A_1 and B_2 sites

Tight-binding model

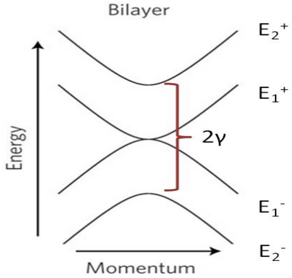
Eigenvalue equation $H\lambda = E\lambda$

$$\lambda_1 = 1\,, \quad \lambda_2 = \frac{\hbar v_F \bar{\kappa}}{E}\,, \quad \lambda_3 = \sigma \frac{\hbar v_F \kappa}{E}\,, \quad \lambda_4 = \sigma\,,$$

where energy satisfies

$$E^2 - |\hbar v_F \kappa|^2 = \sigma \gamma E\,, \qquad \sigma = \pm 1\,.$$

$$E_i^{(\pm)} = \pm \left[(-1)^i \frac{\gamma}{2} + \sqrt{\frac{\gamma^2}{4} + |\hbar v_F \kappa|^2} \right] , i = 1, 2.$$



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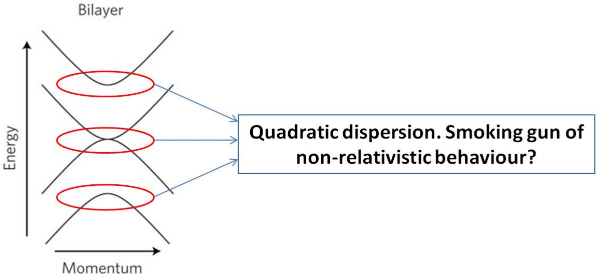

Tight-binding model

Expand in parameter $\epsilon = \frac{\hbar v_F |\kappa|}{\gamma}$

$$\lambda_2 = \pm \frac{\gamma}{\hbar v_F \kappa} + O\left(\epsilon\right) \,, \lambda_3 = \pm \sigma \frac{\gamma}{\hbar v_F \bar{\kappa}} + O\left(\epsilon\right) \,, \quad E_1^{(\pm)} \text{ bands}$$

Tight-binding model

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$$\lambda_2 = \pm \frac{\gamma}{\hbar v_F \kappa} + O(\epsilon) \; , \lambda_3 = \pm \sigma \frac{\gamma}{\hbar v_F \bar{\kappa}} + O(\epsilon) \; , \quad E_1^{(\pm)} \text{ bands}$$

Then for $\tau = 1$

$$\begin{aligned} \chi_1(t) &= e^{\mp i \frac{|\hbar v_F \kappa|^2}{\hbar \gamma} t} (\lambda_1, \sigma \lambda_4)^T \; , \\ \chi_2(t) &= e^{\mp i \frac{|\hbar v_F \kappa|^2}{\hbar \gamma} t} (\lambda_2, \sigma \lambda_3)^T \; , \end{aligned}$$

satisfy the Lévy-Leblond equations

$$\begin{aligned} i\hbar \, \partial_t \chi_2 + i\hbar v_F D \chi_1 = 0 \, , \\ D\chi_2 - i \frac{2 m v_F}{\hbar} \chi_1 = 0 \, , \end{aligned}$$

where $m = \pm \frac{\gamma}{2 v_F^2}$, $D = i \sigma^j k_j$, σ^j Pauli matrices.

Lévy-Leblond equations

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Nonrelativistic Particles and Wave Equations

JEAN-MARC LÉVY-LEBLOND*

Laboratoire de Physique Théorique, Faculté des Sciences de Nice

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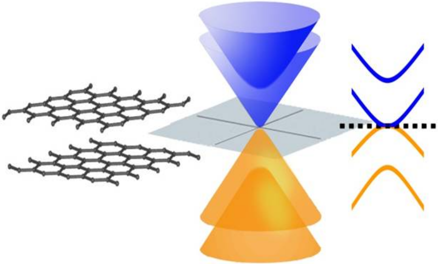
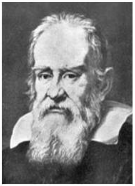
Abstract. This paper is devoted to a detailed study of nonrelativistic particles and their properties, as described by Galilei invariant wave equations, in order to obtain a precise distinction between the specifically relativistic properties of elementary quantum mechanical systems and those which are also shared by nonrelativistic systems. After having emphasized that spin, for instance, is not such a specifically relativistic effect, we construct wave equations for nonrelativistic particles with any spin. Our derivation is based upon the theory of representations of the Galilei group, which define nonrelativistic particles. We particularly study the spin 1/2 case where we introduce a four-component wave equation, the nonrelativistic analogue of the Dirac equation. It leads to the conclusion that the spin magnetic moment, with its Landé factor $g = 2$, is not a relativistic property. More

The wave function Φ is thus a 4-component object, which we write as $\Phi = \binom{\varphi}{\chi}$, φ and χ each being a 2-component function. Our wave equation finally reads:

$$\begin{cases} E\varphi + (\boldsymbol{\sigma}\cdot\mathbf{p})\chi = 0 \\ (\boldsymbol{\sigma}\cdot\mathbf{p})\varphi + 2m\,\chi = 0\,. \end{cases} \qquad (25)$$



Here live Lévy-Leblond quasiparticles



Curvatronics

Lévy-Leblond equations in curved 2D space:

$$(i\hbar\,\partial_t - V)\,\chi_2 + i\hbar v_F\left(\nabla_j - i\frac{e}{\hbar}A_j\right)\sigma^j\chi_1 = 0\,, \quad (*)$$

$$\left(\nabla_j - i\frac{e}{\hbar}A_j\right)\sigma^j\chi_2 - i\frac{2mv_F}{\hbar}\chi_1 = 0\,, \quad (**)$$

$V = V(x,y), A_j = A_j(x,y)$.

Curvatronics

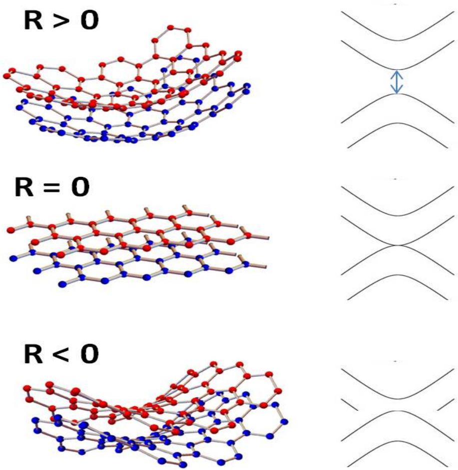
Finding χ_1 in (**) and plugging into (*) gives a Schrödinger equation

$$\left(\mathcal{E} - \frac{g^{ij}}{2m}\Pi_i\Pi_j - V + \frac{e\hbar}{2m}B - \frac{\hbar^2}{8m}R\right)\chi_2 = 0\,.$$

$\Pi_j = -i\hbar\left(\nabla_j - i\frac{e}{\hbar}A_j\right)$ momentum
$B = (\partial_i A_j - \partial_j A_i)\,\sigma^{ij}$ magnetic field
R intrinsic curvature in 2D.

Curvatronics

Positive curvature $\leftrightarrow$ bands separate, insulating
Negative curvature $\leftrightarrow$ bands overlap, metallic



Curvatronics

Model is valid for small changes in energy.

Curvature term contribution is small if $R << \frac{\gamma^2}{\hbar^2 v_F^2}$.

For constant curvature $|R| = \frac{2}{l_R^2}$ this is $l_R >> 1.6nm$. Deformations are smooth on the scale of the hexagonal lattice.

Experimental values can be measured for example with ARPES, $E_A = 10^{-3}eV$. $E_{curv} \geq E_A \leftrightarrow l_R \leq 23nm$, well within typical curvature scale in graphene systems.

Congratulations Gary!
And thank you.