Comparison theorems for causal diamonds

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Summary

1. **Definitions**
2. **The geometry of small causal diamonds**
3. **Comparison theorems for causal diamonds**

   CB, G. Gibbons, and S. Solodukhin, Phys. Rev. D92, 064036, arXiv:1507.03619
4. **An application**

Definitions

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A spacetime is a connected time-oriented Lorentzian manifold $\{M, g\}$

- **The chronological future** of $p \in M$, denoted $I^+(p)$:
  The set of events that can be reached by a future directed timelike curve starting from p.
  For a general subset $S \subset M$, we define $I^+(S) = \cup_{p \in S} I^+(p)$.

- We denote by $\partial I^+(p)$ the boundary of $I^+(p)$.
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- A subset $S \subset M$ is said to be **achronal** if $I^+(S) \cap S = \emptyset$.
- A timelike curve is said to be **past-inextendible** if it has no past endpoint.
- The **future domain of dependence** of an achronal set S, denoted $D^+(S)$, is the set of all $p \in M$ such that every past-inextendible timelike curve through p intersects S.

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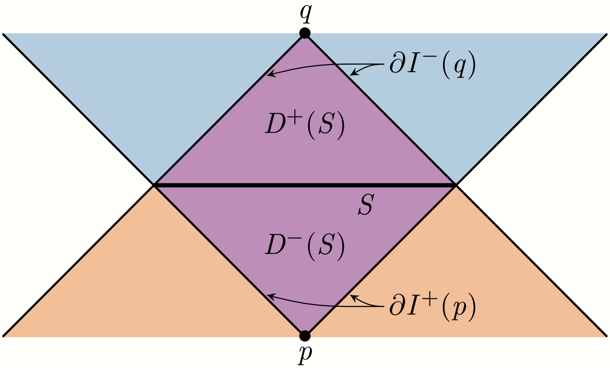
Definitions

Causal diamond

Defined by initial and final events p and q or alternatively by an achronal set S, as a subset of a spacetime of the form:

$$I^+(p) \cap I^-(q) \qquad \text{or} \qquad D^+(S) \cup D^-(S)$$



Definitions

Geometric quantities of interest

- **The** d**-volume** $V(p,q) = \int_{I^+(p)\cap I^-(q)} d^d x \sqrt{-g}$.
- **The spatial volume** $\mathcal{V}(p,q)$ of a hypersurface $(t=0)$ having the intersection $\partial I^+(p) \cap \partial I^-(q)$ as its boundary.
- **The area** $A(p,q)$ of the intersection $\partial I^+(p) \cap \partial I^-(q)$.

The geometry of small causal diamonds

The geometry of small causal diamonds

In curved spacetime, if p and q are sufficiently close, there is a unique time-like geodesic parametrized by proper time τ joining them.

- In small enough causal diamond the metric can be expanded as

$$g_{\mu\nu} = \eta_{\mu\nu} - \frac{1}{3} x^{\alpha} x^{\beta} R_{\mu\alpha\nu\beta}(0) + \cdots$$

  where x^{μ} are RNCs about the center of the diamond $r = 0$.

- The volume of a small causal diamond is

$$V(p,q) = \int_D d^d x \left(1 - \frac{1}{6} x^{\mu} x^{\nu} R_{\mu\nu}(0) + \cdots\right)$$

  where $D = D^+(S) \cup D^-(S)$ in the curved spacetime.

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The geometry of small causal diamonds

The integration over D contains corrections to the Minkowski space-time domain D_M. The integral can be split into two parts $V_1 + V_2$:

$$V(p,q) = \int_{D_M} d^d x \left(1 - \frac{1}{6} x^\mu x^\nu R_{\mu\nu}(0)\right) + \int_{D-D_M} d^d x + \cdots$$

- The evaluation of the first term is straightforward:

$$V_1 = V_M \left(1 - \frac{1}{24(d+1)} \left(R_{00} + \frac{d}{d+2} R\right) \tau^2\right)$$

- The 2^{nd} piece is computed using perturbations of null geodesics:

$$V_2 = \frac{V_M}{24} R_{00} \tau^2$$

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Then the volume of a small causal diamond is:

$$V(p,q) \;=\; V_M(\tau)\left(1+\frac{d}{24(d+1)}\left(R_{00}-\frac{1}{d+2}R\right)\tau^2\;+\;\cdots\right)$$

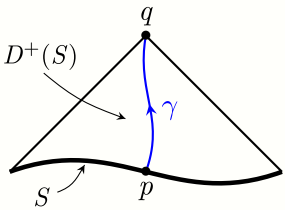
G. Gibbons and S. Solodukhin (2007) also found expressions for the spatial volume and area:

$$\mathcal{V}(p,q) \;=\; \mathcal{V}_M(\tau)\left(1+\frac{d-1}{24(d+1)}\left(R_{00}-\frac{1}{d-1}R\right)\tau^2\;+\;\cdots\right)$$

$$A(p,q) \;=\; A_M(\tau)\left(1+\frac{1}{24(d-1)}\Big(2(d-4)R_{00}-R\Big)\tau^2\;+\;\cdots\right)$$

Extrinsic curvature contributions

- Now consider only a causal cone, i.e. the future or past domain of dependence $D^{\pm}(S)$ of an achronal set S (a hypersurface) with non-vanishing extrinsic curvature K.
- The timelike geodesic γ of duration τ starts at p and ends at q.



- By dimensionality we expect corrections to the volume as:

$$\begin{aligned} 1^{st}\ \text{order} &: \quad K \\ 2^{nd}\ \text{order} &: \quad R, R_{\mu\nu}n^{\mu}n^{\nu}, K^2, \mathrm{Tr}K^2 \end{aligned}$$

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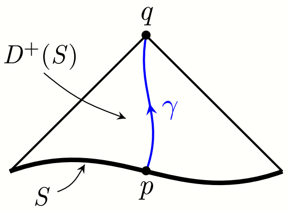

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Extrinsic curvature contributions

The corrections from the extrinsic curvature is found by taking the spacetime to be flat,

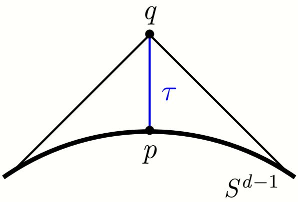
$$V_+(\tau) = V_{flat}\Big(1 + c_0 K\tau + \left(c_1 K^2 + c_2 \mathrm{Tr}K^2\right)\tau^2 + \cdots\Big)$$

There is two methods to find the coefficients c_i:

- Computing the volume between the hypersurface S and the flat cone using RNCs centered at p, [Buck et al. (2015)].
- Computing directly the volume for simple hypersurfaces S (e.g. with constant curvature) and solve for the coefficients c_i, [Jubb (2016)].

Extrinsic curvature contributions

Let's see how to find c_0 by computing the volume of a small causal cone on top of a unit S^{d-1} of extrinsic curvature $K = d - 1$:



- The sphere is parametrized as $t = \sqrt{1 - r^2}$ and the cone as $t = 1 + \tau - r$. They intersect at $r_+(\tau) = \frac{1}{2}(1 + \tau - \sqrt{1 - 2\tau - \tau^2})$.
- For a small cone $\tau \ll 1$ one finds:

$$V_+(\tau) = \Omega_{d-2} \int_0^{r_+(\tau)} dr\, r^{d-2} \int_{\sqrt{1-r^2}}^{1+\tau-r} dt$$
$$\simeq V_{flat} \left(1 + \frac{d(d-1)}{2(d+1)}\tau + \cdots\right)$$

Then we identify $c_0 = \dfrac{d}{2(d+1)}$.

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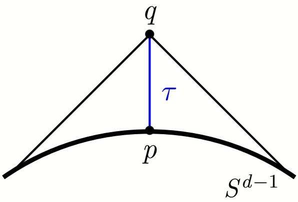

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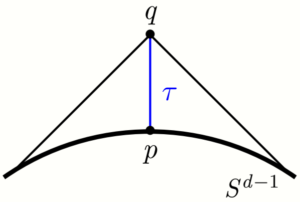

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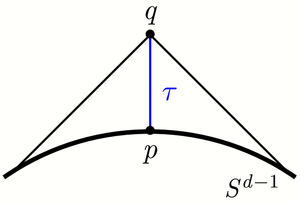

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Extrinsic curvature contributions

Playing the same game with a family of hypersurfaces S one finds up to $O(\tau^2)$ expressions for the d-volume, spatial volume and area of a small causal cone:

- **The d-volume** [Jubb (2016)]

$$V_+(\tau) = V_{flat}\left(1 + \frac{d}{2(d+1)}K\tau + \frac{d}{4(d+1)}\left(\frac{1}{2}K^2 + \mathrm{Tr}K^2\right)\tau^2 + \cdots\right)$$

- **The spatial volume**

$$\mathcal{V}_+(\tau) = \mathcal{V}_{flat}\left(1 + \frac{1}{2}K\tau + \frac{d+2}{4(d+1)}\left(\frac{1}{2}K^2 + \frac{d}{d+2}\mathrm{Tr}K^2\right)\tau^2 + \cdots\right)$$

- **The area**

$$A_+(\tau) = A_{flat}\left(1 + \frac{d-2}{2(d-1)}K\tau + \frac{d-2}{4(d-1)}\left(\frac{1}{2}K^2 + \mathrm{Tr}K^2\right)\tau^2 + \cdots\right)$$

Comparison theorems for causal diamonds

Metric and asymptotic conditions

We consider a **spherically symmetric** metric in d dimensions:

$$ds^2 = -f(r)\mathrm{e}^{2\gamma(r)}\,dt^2 + \frac{dr^2}{f(r)} + r^2\,d\Omega^2_{d-2}\,,$$

$$\text{with} \quad f(r) = 1 - \frac{2m(r)}{r^{d-3}} > 0\,.$$

The Einstein's equations $R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = (d-2)\kappa_d\,T_{\mu\nu}$ give:

$$\frac{dm(r)}{dr} = \kappa_d\,r^{d-2}\,T_{\hat{t}\hat{t}}\,,$$

$$\frac{d\gamma(r)}{dr} = \kappa_d\,r\,f(r)^{-1}\left(T_{\hat{t}\hat{t}} + T_{\hat{r}\hat{r}}\right)\,.$$

Metric and asymptotic conditions

- We shall assume some **energy positivity conditions**:

$$T_{\hat{t}\hat{t}} \geq 0 \quad \text{and} \quad T_{\hat{r}\hat{r}} \geq 0$$

- **Asymptotic flatness conditions**:

$$\lim_{r\to\infty} f(r) = 1 \quad \text{and} \quad \lim_{r\to\infty} \gamma(r) = 0$$

$$f(0) = 1 \quad \text{and} \quad \gamma(0) < 0$$

$\Longrightarrow \quad m(r) \geq 0\,,\ \gamma(r) \leq 0\,,$ both monotonic increasing.

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Monotonicity of the redshift

The redshift $z(r)$ is defined as

$$\frac{1}{1+z} = \sqrt{-g_{tt}(r)}, \qquad \text{with} \qquad -g_{tt} = f(r)e^{2\gamma(r)}$$

The energy conditions and asymptotics of $m(r)$ and $\gamma(r)$ yield

$$-\frac{dg_{tt}(r)}{dr} = \frac{2e^{2\gamma(r)}}{r^{d-2}}\Big((d-3)m(r) + \kappa_d\, r^{d-1}\, T_{\hat{r}\hat{r}}\Big) \geq 0$$

and $e^{2\gamma_0} \leq f(r)e^{2\gamma(r)} \leq 1$.

Then the **redshift** $z(r)$ **is a monotonic decreasing function of** r,

$$z_c \geq z(r) \geq 0$$

where $z_c = z(r=0) = e^{-\gamma_0} - 1$ is the redshift at the center.

The causal diamond

The diamond is determined by two events p and q joined by a **timelike** geodesic (observer at rest at $r = 0$) of **invariant duration** τ (T in t time).

- Relation between t time and proper-time τ:

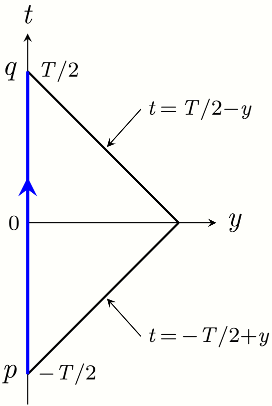
$$T = (1 + z_c)\,\tau$$

- Introduce tortoise coordinates y:

$$y = \int_0^r dr' \, \frac{e^{-\gamma(r')}}{f(r')}$$

- The value of $r(T/2)$ is found from:

$$T/2 = \int_0^{r(T/2)} dr' \, \frac{e^{-\gamma(r')}}{f(r')}$$



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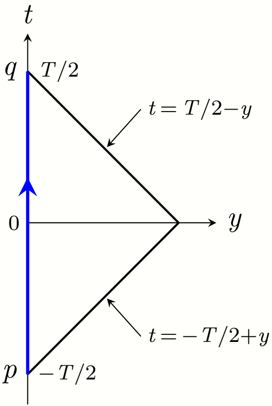

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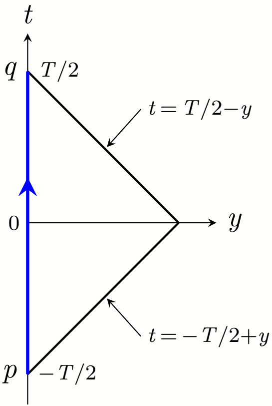

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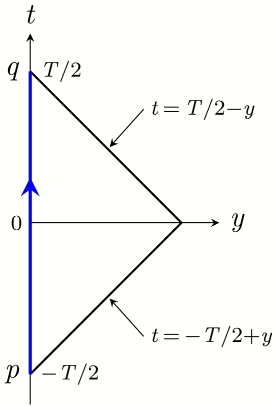

Volume, spatial volume and area

- The **volume** of this diamond is:

$$V(\tau) = 2\Omega_{d-2}\int_0^{T/2} dt \int_0^{T/2-t} dy\, r^{d-2}(y) f(y) e^{2\gamma(y)}$$

- The **spatial volume** is:

$$\mathcal{V}(\tau) = \Omega_{d-2}\int_0^{r(T/2)} dr' \,\frac{r'^{d-2}}{\sqrt{f(r')}}$$

- The **area** is:

$$A(\tau) = \Omega_{d-2} r^{d-2}(T/2)$$

Comparison theorems

Monotonicity and boundary conditions of $\gamma(r)$ and $m(r)$ yield

$$\tau \leq r(T) \leq (1+z_c)\tau\,,$$

Comparison theorems [CB, G. Gibbons, S Solodukhin (2015)]

- **The d-volume** satisfies

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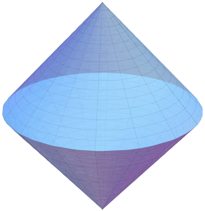
Generalization to spacetime with horizon

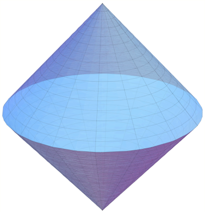
- **The “usual” causal diamond:**
  - The achronal set S is a ball in a spacelike hypersurface orthogonal to the time-like geodesic joining p and q.
  - p and q are points.



- **Generalized causal diamond:**
  - The achronal set is a solid annulus of the form $I \times S^{d-2}$, where I is the interval in the radial direction.
  - p and q are S^{d-2} and I is given by $y_0 - \frac{1}{2}\tau \leq y_0 + \frac{1}{2}\tau$.
  - Can be applied if a **black hole** is present.

Generalization to spacetime with horizon

- **The "usual" causal diamond:**
  - The achronal set S is a ball in a spacelike hypersurface orthogonal to the time-like geodesic joining p and q.
  - p and q are points.



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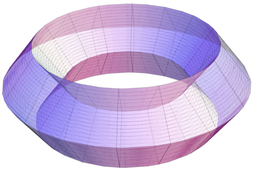

Causal diamond in Schwarzschild spacetime

We consider the d-dimensional Schwarzschild spacetime:

$$ds^2_{Sch^d} = -f_d(r)\,dt^2 + f_d(r)^{-1}\,dr^2 + r^2\,d\Omega^2_{d-2}\,, \qquad f_d(r) = 1 - \left(\frac{r_s}{r}\right)^{d-3}$$

- Time-like geodesic joining p and q is a **round trip** from r_0 to r_{max} and then back to r_0.
- Introduce tortoise coordinates y:

$$y(r) - y(r_0) = \int_{r_0}^{r} \frac{dr'}{f_d(r')} = T/2$$

$$\Rightarrow \quad r(T \gg 1) \simeq T + \frac{r_s}{d-4}\left(\frac{r_s}{T}\right)^{d-4} + \gamma_d(r_0)$$

- The volume reads:

$$V_{Sch^d}(\tau, r_0) = 2\Omega_{d-2} \int_0^{T/2} dt \int_{-T/2+t}^{T/2-t} dy\, r^{d-2}(y) f_d(y)$$

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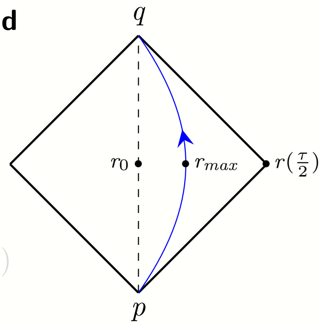

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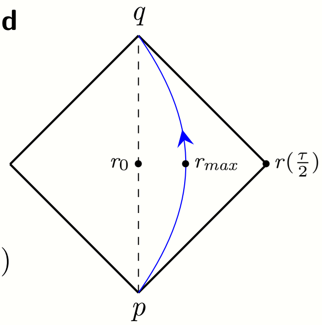

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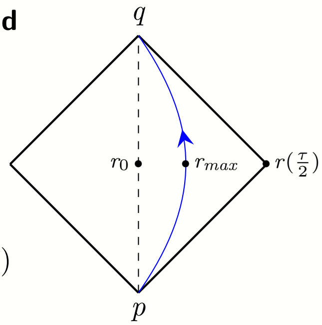

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Proper time of radial timelike geodesics

Timelike radial geodesic equations ($E^2 = f_d(r_{max})$):

$$\frac{dt}{d\tau} = \frac{E}{f_d(r)}, \qquad (1)$$

$$\left(\frac{dr}{d\tau}\right)^2 = E^2 - f_d(r). \qquad (2)$$

Equation (2) gives r_{max} as a function of τ:

$$r_{max}(\tau) \simeq r_s \left(\frac{\tau}{2 b_d r_s}\right)^{2/(d-1)}, \qquad \tau \gg 1$$

Combined with (1) one finds $T(\tau)$:

$$T/2 \simeq \tau/2 + \frac{\alpha_d\, r_s}{5-d}\left(\frac{\tau}{2r_s}\right)^{\frac{5-d}{d-1}} + \beta_d(r_0)$$

where α_d, $\beta_d(r_0) > 0$.

Volume of generalized causal diamond

- Volume in $d = 4$ dimensions:

$$V_{Sch^4}(\tau, r_0) \simeq \frac{\pi}{24}\tau^4 + \frac{5\pi r_s}{6}\left(\frac{\pi^2}{8r_s}\right)^{1/3}\tau^{10/3} + \cdots > V_{M^4}(\tau)$$

- Volume in $d = 5$ dimensions:

$$V_{Sch^5}(\tau, r_0) \simeq \frac{\pi}{160}\tau^5 + \frac{\pi^2}{32} r_s \tau^4 \ln \tau / r_s + \cdots > V_{M^5}(\tau)$$

- Volume in $d > 5$ dimensions:

$$V_{Sch^d}(\tau, r_0) \simeq V_{M^d}(\tau)\left(1 + 2d\frac{\sigma_d(r_0)}{\tau} + \cdots\right) > V_{M^d}(\tau)$$

Inequalities involving volumes in flat spacetime M $\forall d$

$$V_{Sch^d}(\tau, r_0) > V_{M^d}(\tau, r_0) > V_{M^d}(\tau), \qquad V_{M^d}(\tau, r_0) \simeq V_{M^d}(\tau) \times \left(1 + 2d\frac{r_0}{\tau} + \cdots\right)$$

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An application

Comparison theorems

Comparison theorems [CB, G. Gibbons, S Solodukhin (2015)]

- **The d-volume** satisfies

$$V_M(\tau) \leq V(\tau) \leq (1+z_c)^d \, V_M(\tau)\,, \qquad V_M(\tau) = \frac{2\Omega_{d-2}}{d(d-1)}(\tau/2)^d$$

- **The spatial volume** satisfies

$$\mathcal{V}_M(\tau) \leq \mathcal{V}(\tau) \leq (1+z_c)^{d-1} \, \mathcal{V}_M(\tau)\,, \qquad \mathcal{V}_M(\tau) = \frac{\Omega_{d-2}}{d-1}(\tau/2)^{d-1}$$

- **The area** satisfies

$$A_M(\tau) \leq A(\tau) \leq (1+z_c)^{d-2} \, A_M(\tau)\,, \qquad A_M(\tau) = \Omega_{d-2}(\tau/2)^{d-2}$$

A conjecture

The area of a causal diamond satisfies

$$A_M(\tau) \leq A(\tau) \leq (1+z_c)^{d-2} A_M(\tau)$$

Now consider a surface Σ associated with a causal diamond of duration τ, in a curved spacetime $\mathcal{M}$ and in Minkowski spacetime M.

Reformulating the inequalities on the area in terms of entanglement entropy we conjecture:

$$S_\Sigma(M,\tau) \leq S_\Sigma(\mathcal{M},\tau) \leq (1+z_c)^{d-2} S_\Sigma(M,\tau)$$

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Thank you!