A New Approach for Designing Moving-Water Equilibria Preserving Schemes for the Shallow Water Equations

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joint work with
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Systems of Balance Laws

$$\boldsymbol{U}_t + \boldsymbol{f}(\boldsymbol{U})_x + \boldsymbol{g}(\boldsymbol{U})_y = \boldsymbol{S}(\boldsymbol{U})$$

Examples:

- Gas dynamics with pipe-wall friction
- Euler equations with gravity/friction
- shallow water equations with Coriolis forces

Applications:

- astrophysical and atmospheric phenomena in many fields including supernova explosions
- (solar) climate modeling and weather forecasting

$$\boldsymbol{U}_t + \boldsymbol{f}(\boldsymbol{U})_x + \boldsymbol{g}(\boldsymbol{U})_y = \frac{1}{\varepsilon}\boldsymbol{S}(\boldsymbol{U})$$

Examples:

- low Mach number compressible flows
- low Froude number shallow water flows
- diffusive relaxation in kinetic models

Applications:

- various two-phase flows such as bubbles in water
- unmostly incompressible flows with regions of high compressibility such as underwater explosions
- atmospheric flows

Systems of Balance Laws

$$\boldsymbol{U}_t + \boldsymbol{f}(\boldsymbol{U})_x + \boldsymbol{g}(\boldsymbol{U})_y = \boldsymbol{S}(\boldsymbol{U}) \quad \text{or} \quad \boldsymbol{U}_t + \boldsymbol{f}(\boldsymbol{U})_x + \boldsymbol{g}(\boldsymbol{U})_y = \frac{1}{\varepsilon}\boldsymbol{S}(\boldsymbol{U})$$

- Challenges: certain structural properties of these hyperbolic problems (conservation or balance law, equilibrium state, positivity, assymptotic regimes, etc.) are essential in many applications;

- Goal: to design numerical methods that are not only consistent with the given PDEs, but
  - preserve the structural properties at the discrete level – **well-balanced numerical methods**
  - remain accurate and robust in certain asymptotic regimes of physical interest – **asymptotic preserving numerical methods**

[P. LeFloch; 2014]

Well-Balanced (WB) Methods

$$\boldsymbol{U}_t + \boldsymbol{f}(\boldsymbol{U})_x + \boldsymbol{g}(\boldsymbol{U})_y = \boldsymbol{S}(\boldsymbol{U})$$

- In many physical applications, solutions of the system are small perturbations of the steady states;
- These perturbations may be smaller than the size of the truncation error on a coarse grid;
- To overcome this difficulty, one can use very fine grid, but in many physically relevant situations, this may be unaffordable;

Goal:

- to design a well-balanced numerical method, that is, the method which is capable of exactly preserving some steady state solutions;
- perturbations of these solutions will be resolved on a coarse grid in a non-oscillatory way.

Finite-Volume Methods – 1-D

$$\boldsymbol{U}_t + \boldsymbol{f}(\boldsymbol{U})_x = \boldsymbol{S}$$

- $\overline{\boldsymbol{U}}_k^n \approx \dfrac{1}{\Delta y} \displaystyle\int\limits_{C_k} \boldsymbol{U}(y, t^n)\, dy$: cell averages over $C_j := (x_{j-\frac{1}{2}}, x_{j+\frac{1}{2}})$

- Semi-discrete FV method:

$$\frac{d}{dt}\overline{\boldsymbol{U}}_j(t) = -\frac{\boldsymbol{\mathcal{F}}_{j+\frac{1}{2}}(t) - \boldsymbol{\mathcal{F}}_{j-\frac{1}{2}}(t)}{\Delta x} + \overline{\boldsymbol{S}}_j$$

$\boldsymbol{\mathcal{F}}_{j+\frac{1}{2}}(t)$: numerical fluxes

$\overline{\boldsymbol{S}}_j$: quadrature approximating the corresponding source terms

- Central-Upwind (CU) Scheme:

[Kurganov, Lin, Noelle, Petrova, Tadmor, et al.; 2000–2007]

$$\{\overline{\boldsymbol{U}}_j(t)\} \to \widetilde{\boldsymbol{U}}(\cdot,t) \to \left\{\boldsymbol{U}_j^{\mathrm{E,W}}(t)\right\} \to \left\{\boldsymbol{\mathcal{F}}_{j+\frac{1}{2}}(t)\right\} \to \{\overline{\boldsymbol{U}}_j(t+\Delta t)\}$$

(Discontinuous) piecewise-linear reconstruction:

$$\widetilde{\boldsymbol{U}}(y,t) := \overline{\boldsymbol{U}}_j(t) + (\boldsymbol{U}_x)_j(x - x_j), \quad x \in C_j$$

It is conservative, second-order accurate, and non-oscillatory provided the slopes, $\{(\boldsymbol{U}_y)_k\}$, are computed by a nonlinear limiter

Example — Generalized Minmod Limiter

$$(\boldsymbol{U}_y)_j = \mathrm{minmod}\left(\theta\frac{\overline{\boldsymbol{U}}_j - \overline{\boldsymbol{U}}_{j-1}}{\Delta x}, \frac{\overline{\boldsymbol{U}}_{j+1} - \overline{\boldsymbol{U}}_{j-1}}{2\Delta x}, \theta\frac{\overline{\boldsymbol{U}}_{j+1} - \overline{\boldsymbol{U}}_j}{\Delta x}\right)$$

where

$$\mathrm{minmod}(z_1, z_2, ...) := \begin{cases} \min_j\{z_j\}, & \text{if } z_j > 0 \ \forall j, \\ \max_j\{z_j\}, & \text{if } z_j < 0 \ \forall j, \\ 0, & \text{otherwise}, \end{cases}$$

and $\theta \in [1, 2]$ is a constant

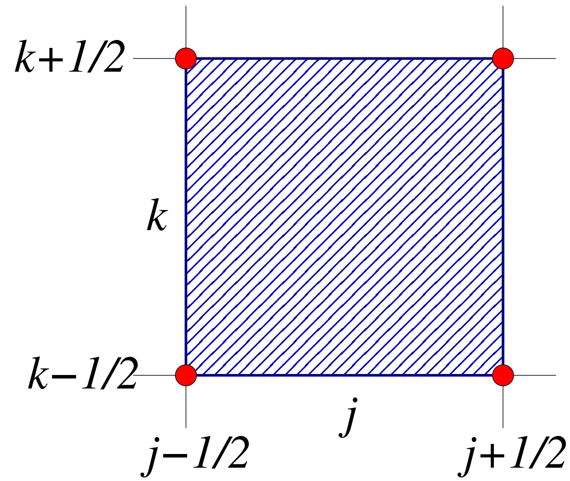
$$\{\overline{\boldsymbol{U}}_j(t)\} \to \widetilde{\boldsymbol{U}}(\cdot,t) \to \left\{\boldsymbol{U}_j^{\mathrm{E,W}}(t)\right\} \to \left\{\boldsymbol{\mathcal{F}}_{j+\frac{1}{2}}(t)\right\} \to \{\overline{\boldsymbol{U}}_j(t+\Delta t)\}$$

$\boldsymbol{U}_j^{\mathrm{E}}$ and $\boldsymbol{U}_j^{\mathrm{W}}$ are the point values at $x_{j+\frac{1}{2}}$ and $x_{j-\frac{1}{2}}$:

$$\widetilde{\boldsymbol{U}}(y,t) = \overline{\boldsymbol{U}}_j + (\boldsymbol{U}_x)_j(x - x_j), \quad x \in C_j$$

$$\boldsymbol{U}_j^{\mathrm{E}} := \overline{\boldsymbol{U}}_j + \frac{\Delta x}{2}(\boldsymbol{U}_x)_j$$

$$\boldsymbol{U}_j^{\mathrm{W}} := \overline{\boldsymbol{U}}_j - \frac{\Delta x}{2}(\boldsymbol{U}_x)_j$$



$$\{\overline{\boldsymbol{U}}_j(t)\} \to \widetilde{\boldsymbol{U}}(\cdot,t) \to \left\{\boldsymbol{U}_j^{\mathrm{E,W}}(t)\right\} \to \left\{\boldsymbol{\mathcal{F}}_{j+\frac{1}{2}}(t)\right\} \to \{\overline{\boldsymbol{U}}_j(t+\Delta t)\}$$

$$\frac{d}{dt}\overline{\boldsymbol{U}}_j = -\frac{\boldsymbol{\mathcal{F}}_{j+\frac{1}{2}} - \boldsymbol{\mathcal{J}}_{j-\frac{1}{2}}}{\Delta x} + \overline{\boldsymbol{S}}_j$$

where

$$\boldsymbol{\mathcal{F}}_{j+\frac{1}{2}} = \frac{a^+_{j+\frac{1}{2}}\boldsymbol{f}(\boldsymbol{U}_j^{\mathrm{E}}) - a^-_{j+\frac{1}{2}}\boldsymbol{f}(\boldsymbol{U}_{j+1}^{\mathrm{W}})}{a^+_{j+\frac{1}{2}} - a^-_{j+\frac{1}{2}}} + \alpha_{j+\frac{1}{2}}\left(\boldsymbol{U}_{j+1}^{\mathrm{W}} - \boldsymbol{U}_j^{\mathrm{W}}\right)$$

$$\alpha_{j+\frac{1}{2}} = \frac{a^+_{j+\frac{1}{2}}a^-_{j+\frac{1}{2}}}{a^+_{j+\frac{1}{2}} - a^-_{j+\frac{1}{2}}}$$

$$a^+_{j+\frac{1}{2}} = \max\left\{\lambda(\boldsymbol{U}_j^{\mathrm{E}}), \lambda(\boldsymbol{U}_{j+1}^{\mathrm{W}}), 0\right\}, \quad a^-_{j+\frac{1}{2}} = \min\left\{\lambda(\boldsymbol{U}_j^{\mathrm{E}}), \lambda(\boldsymbol{U}_{j+1}^{\mathrm{W}}), 0\right\}$$

2-D extension is dimension-by-dimension

Non Well-Balanced Property – Example

$$\begin{cases} h_t + q_x = 0, \\ q_t + f_2(h,q)_x = -s(h,q) \end{cases}$$

For steady-state solution: $q = \text{Const}$ and $h = h(x)$

Implementing the CU scheme results in

$$\frac{d\bar{h}_j}{dt} = -\frac{1}{\Delta x}\left[\frac{a^+_{j+\frac{1}{2}} q^{\rm E}_j - a^-_{j+\frac{1}{2}} q^{\rm W}_{j+1}}{a^+_{j+\frac{1}{2}} - a^-_{j+\frac{1}{2}}} + \alpha_{j+\frac{1}{2}}(h^{\rm W}_{j+1} - h^{\rm E}_j)\right.$$

$$\left. - \frac{a^+_{j-\frac{1}{2}} q^{\rm E}_{j-1} - a^-_{j-\frac{1}{2}} q^{\rm W}_j}{a^+_{j-\frac{1}{2}} - a^-_{j-\frac{1}{2}}} + \alpha_{j-\frac{1}{2}}(h^{\rm W}_j - h^{\rm E}_{j-1})\right] \neq 0$$

- The steady state would not be preserved at the discrete level;
- This would also true for the first-order version of the scheme;
- For smooth solutions, the balance error is expected to be of order $(\Delta x)^2$, but a coarse grid solution may contain large spurious waves.

Well-Balanced Methods

1-D 2×2 Systems of Balance Laws

$$\begin{cases} h_t + f_1(h,q)_x = 0, \\ q_t + f_2(h,q)_x = -s(h,q), \end{cases}$$

Steady state solution:

$$f_1(h,q)_x \equiv 0, \quad f_2(h,q)_x + s(h,q) \equiv 0$$

or

$$K := f_1(h,q) \equiv \text{Const},$$

$$L := f_2(h,q) + \int^x s(h,q)d\xi \equiv \text{Const} \qquad \forall x, t$$

Numerical Challenges : to exactly balance the flux and source terms, i.e., to exactly preserve the steady states.

How to design a well-balanced scheme?

Well-Balanced Scheme

$$\begin{cases} h_t + f_1(h,q)_x = 0, \\ q_t + f_2(h,q)_x = -s(h,q) \end{cases}$$

- Incorporate the source term into the flux:

$$\begin{cases} h_t + f_1(h,q)_x = 0, \\ q_t + \left(f_2(h,q)_x + R\right)_x = 0, \end{cases} \qquad R := \int^x s(h,q)\,d\xi$$

- Rewrite

$$\begin{cases} h_t + K_x = 0, \\ q_t + L_x = 0 \end{cases}$$

where

$$K := f_1(h,q), \qquad L := f_2(h,q)_x + R$$

- Define

conservative variables $\boldsymbol{U} = (h,q)^T$

equilibrium variables $\mathbf{W} := (K,L)^T$

Well-Balanced Scheme

$$\boldsymbol{U}_t + \boldsymbol{f}(\boldsymbol{U})_x = \boldsymbol{0}$$

$$\boldsymbol{U} = \begin{pmatrix} h \\ q \end{pmatrix}, \quad \boldsymbol{f}(\boldsymbol{U}) = \boldsymbol{W} := \begin{pmatrix} K \\ L \end{pmatrix}$$

Semi-discrete FV method:

$$\frac{d}{dt}\overline{\boldsymbol{U}}_j(t) = -\frac{\boldsymbol{\mathcal{F}}_{j+\frac{1}{2}}(t) - \boldsymbol{\mathcal{F}}_{j-\frac{1}{2}}(t)}{\Delta x}$$

Two major modifications:

- Well-balanced reconstruction – *performed on the equilibrium rather than conservative variables*:

$$\{\overline{\boldsymbol{U}}_j(t)\} \to \widetilde{\boldsymbol{U}}(\cdot,t) \to \left\{\mathbf{W}_j^{\mathrm{E,W}}(t)\right\} \to \left\{\boldsymbol{U}_j^{\mathrm{E,W}}(t)\right\} \to \left\{\boldsymbol{\mathcal{F}}_{j+\frac{1}{2}}(t)\right\} \to \{\overline{\boldsymbol{U}}_j(t+\Delta t)\}$$

- Well-balanced evolution

Well-Balanced Reconstruction

Given: $\overline{\boldsymbol{U}}_j(t) = (\overline{h}_j, \overline{q}_j)^T$ – cell averages

Need: $\mathbf{W}_j^{\mathrm{E,W}} = (K_j^{\mathrm{E,W}}, L_j^{\mathrm{E,W}})^T$ – point values, where

$$K := f_1(h,q), \quad L := f_2(h,q)_x + R, \quad R := \int^x s(h,q)d\xi$$

- Compute $R_j = \displaystyle\int^{x_j} s(h,q)d\xi$ by the midpoint quadrature rule and using the following recursive relation:

$$R_{1/2} \equiv 0, \quad R_j = \frac{1}{2}(R_{j-\frac{1}{2}} + R_{j+\frac{1}{2}}),$$
$$R_{j+\frac{1}{2}} = R(x_{j+\frac{1}{2}}) = R_{j-\frac{1}{2}} + \Delta x\, s(x_j, \overline{h}_j, \bar{q}_j)$$

- Compute the point values of K and L at x_j from the cell averages, $\overline{h}_j$ and $\bar{q}_j$:

$$K_j = f_1(\overline{h}_j, \bar{q}_j), \qquad L_j = f_2(\overline{h}_j, \bar{q}_j) + R_j$$

Well-Balanced Reconstruction

- Apply the minmod reconstruction procedure to $\{K_j, L_j\}$ and obtain the point values at the cell interfaces:

$$K_j^{\mathrm{E}} = K_j + \frac{\Delta x}{2}(K_x)_j, \quad L_j^{\mathrm{E}} = L_j + \frac{\Delta x}{2}(L_x)_j,$$
$$K_j^{\mathrm{W}} = K_j - \frac{\Delta x}{2}(K_x)_j, \quad L_j^{\mathrm{W}} = L_j - \frac{\Delta x}{2}(L_x)_j$$

- Finally, equipped with the values of $K_j^{\mathrm{E,W}}$, $L_j^{\mathrm{E,W}}$ and $R_{j\pm\frac{1}{2}}$, solve

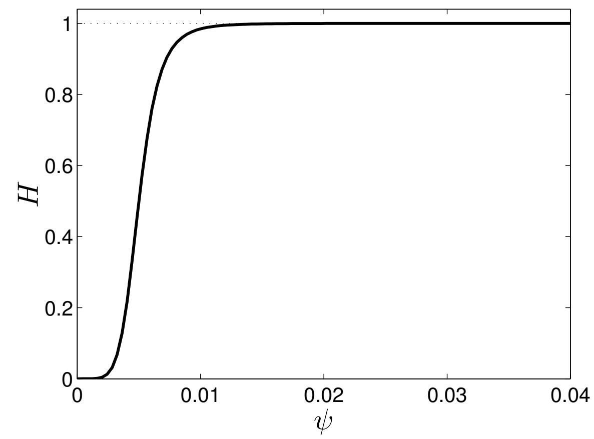
$$K_j^{\mathrm{E}} = f_1(h_j^{\mathrm{E}}, q_j^{\mathrm{E}}), \qquad L_j^{\mathrm{E}} = f_2(h_j^{\mathrm{E}}, q_j^{\mathrm{E}}) + R_{j+\frac{1}{2}},$$
$$K_j^{\mathrm{W}} = f_1(h_j^{\mathrm{W}}, q_j^{\mathrm{W}}), \quad L_j^{\mathrm{W}} = f_2(h_j^{\mathrm{W}}, q_j^{\mathrm{W}}) + R_{j-\frac{1}{2}}$$

for $\boldsymbol{U}_j^{\mathrm{E,W}} = (h_j^{\mathrm{E,W}}, q_j^{\mathrm{E,W}})^T$.

Well-Balanced Evolution

$$\frac{d}{dt}\overline{\boldsymbol{U}}_j = -\frac{\boldsymbol{\mathcal{F}}_{j+\frac{1}{2}} - \boldsymbol{\mathcal{F}}_{j-\frac{1}{2}}}{\Delta x}$$

where



$$\mathcal{F}^{(1)}_{j+\frac{1}{2}} = \frac{a^{+}_{j+\frac{1}{2}} K^{\mathrm{E}}_{j} - a^{-}_{j+\frac{1}{2}} K^{\mathrm{W}}_{j+1}}{a^{+}_{j+\frac{1}{2}} - a^{-}_{j+\frac{1}{2}}}$$

$$+ \alpha_{j+\frac{1}{2}}(h^{\mathrm{W}}_{j+1} - h^{\mathrm{E}}_{j})\, \mathcal{H}\left(\frac{|K_{j+1} - K_j|}{\Delta x} \cdot \frac{|\Omega|}{\max_j} K_j, K_{j+1}\}\right),$$

$$\mathcal{F}^{(2)}_{j+\frac{1}{2}} = \frac{a^{+}_{j+\frac{1}{2}} L^{\mathrm{E}}_{j} - a^{-}_{j+\frac{1}{2}} L^{\mathrm{W}}_{j+1}}{a^{+}_{j+\frac{1}{2}} - a^{-}_{j+\frac{1}{2}}}$$

$$+ \alpha_{j+\frac{1}{2}}(q^{\mathrm{W}}_{j+1} - q^{\mathrm{E}}_{j})\, \mathcal{H}\left(\frac{|L_{j+1} - L_j|}{\Delta x} \cdot \frac{|\Omega|}{\max_j\{L_j, L_{j+1}\}}\right),$$

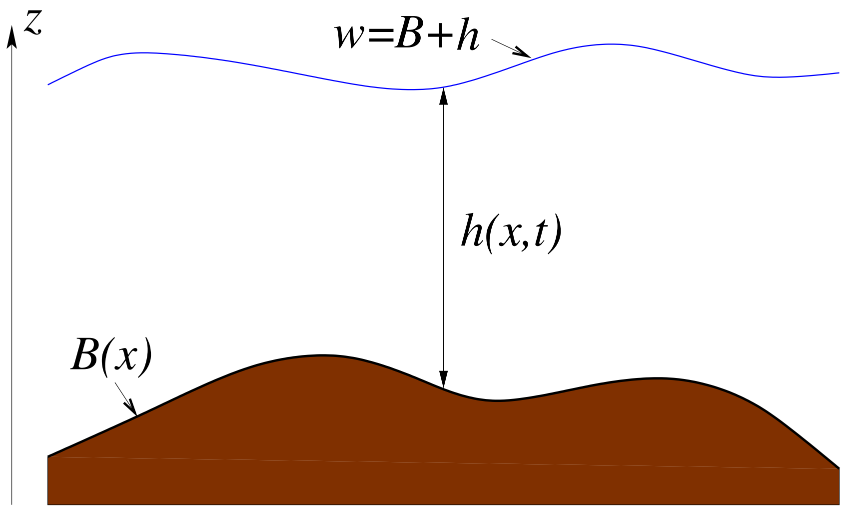
Proof of the Well-Balanced Property

Theorem. *The central-upwind semi-discrete schemes coupled with the well-balanced reconstruction and evolution is well-balanced in the sense that it preserves the corresponding steady states exactly.*

1-D Saint-Venant System of Shallow Water with Friction

$$\begin{cases} h_t + q_x = 0 \\ q_t + \left(hu^2 + \dfrac{g}{2}h^2\right)_x = -ghB_x - g\dfrac{n^2}{h^{7/3}}|q|q \end{cases}$$

- h – water depth
- u – velocity
- $q := hu$ – discharge
- $B(x)$ – bottom elevation
- g – the constant gravitational acceleration
- n – Manning friction coefficient.



Shallow Water Equations

$$\begin{cases} h_t + q_x = 0 \\ q_t + \left(hu^2 + \dfrac{g}{2}h^2\right)_x = -ghB_x - g\dfrac{n^2}{h^{7/3}}|q|q \end{cases}$$

- **Well-balanced scheme** should exactly balance the flux and source terms so that the steady states are preserved:
  - Moving Steady-state solutions (no friction $n \equiv 0$):

$$q = \text{Const}, \quad \frac{u^2}{2} + g(h+B) = \text{Const}$$

  - Stationary steady-state solutions (lake at rest):

$$u = 0, \quad h + B = \text{Const}$$

Well-Balanced Methods – Some References

- Shallow water models (preserving "lake at rest" steady states):
  - LeVeque (1998) – incorporating the source term into the Riemann solver
  - Jin (2001) – well-balanced source term averaging
  - Perthame, Simeoni (2001) – kinetic scheme
  - Kurganov, Levy (2002) – central-upwind scheme
  - Gallouët, Hérard, Seguin (2003) – Roe-type scheme
  - Audusse, Bouchut, Bristeau, Klein, Perthame (2004) – hydrostatic reconstruction
  - Russo (2005) staggered central scheme
  - Xing, Shu (2005, 2006) – WENO schemes
  - Noelle, Pankratz, Puppo, Natvig (2006) – high-order schemes
  - Lukácová-Medvidová, Noelle, Kraft (2007) FVEG scheme
  - Berthon, Marche (2008) – relaxation schemes
  - Fjordholm, Mishra, Tadmor (2008, 2011) energy stable schemes
  - Abgrall, Audusse, Bristeau, Castro, Chertock, Dawson, Donat, Epshteyn, George, Karni, Klingenberg, Mohammadian, Parés, Ricchiuto, ...

Well-Balanced Methods – Some References

- Shallow water models (preserving moving steady states):
  - Noelle, Shu, Xing (2007, 2009, 2011) – WENO schemes
  - Russo, Khe (2009, 2010) – staggered central schemes
  - Xing (2014) – discontinuous Galerkin method
  - Y. Chen, A. Kuragnov (2016) – central-upwind scheme
  - Y. Chen, A. Chertock, M. Herty, A. Kurganov (2017; preprint) – central-upwind scheme

- Shallow water models (positivity preserving schemes):
  - Perthame, Simeoni (2001) – kinetic scheme
  - Audusse, Bouchut, Bristeau, Klein, Perthame (2004) – hydrostatic reconstruction
  - Kurgnov, Petrova (2007) – central-upwind scheme with continuous piecewise linear bottom reconstruction
  - Berthon, Marche (2008) – relaxation schemes
  - Bollermann, Noelle, Lukáčová-Medvid'ová (2011) – special time-quadrature for the fluxes
  - Bollermann, Chen, Kurganov, Noelle (2013): well-balanced reconstruction of wet/dry fronts

Moving Steady States with Friction

$$\begin{cases} \cancel{h_t} + q_x = 0 \\ \cancel{q_t} + \left(hu^2 + \frac{g}{2}h^2\right)_x = -ghB_x - g\frac{n^2}{h^{7/3}}|q|q \end{cases}$$

We incorporate the source term in the discharge equation into its flux term:

$$\begin{cases} h_t + q_x = 0, \\ q_t + \left(\frac{q^2}{h} + \frac{g}{2}h^2 + R\right)_x = 0 \end{cases}$$

General (moving-water) steady state can be expressed in terms of K and L:

$$q \equiv \text{Const}, \quad K \equiv \text{Const}$$

where

$$K := \frac{q^2}{h} + \frac{g}{2}h^2 + R$$

$$R(x,t) := g\int^x \left[h(\xi,t)B_x(\xi) + \frac{n^2}{h^{7/3}(\xi)}|q(\xi)|q(\xi)\right]d\xi$$

Well-Balanced Algorithm

Given: $\overline{\boldsymbol{U}}_j(t) = (\overline{h}_j, \overline{q}_j)^T$ – cell averages

- Compute equilibrium variables $(\bar{q}_j, K_j)^T$ at x_j from the above cell averages:

$$\bar{q}_j, \quad K_j = \frac{\bar{q}_j^{\,2}}{\overline{h}_j} + \frac{g}{2}\overline{h}_j^{\,2} + \frac{R_{j+\frac{1}{2}} + R_{j-\frac{1}{2}}}{2},$$

  where

$$R(x_{j+\frac{1}{2}}, t) \approx R_{j+\frac{1}{2}} := g \sum_{m=j_\ell}^{j} \left\{ \overline{h}_m (B_{m+\frac{1}{2}} - B_{m-\frac{1}{2}}) + \frac{n^2}{\overline{h}_m^{\,7/3}} |\bar{q}_m| \bar{q}_m \Delta x \right\}$$

$$= R_{j-\frac{1}{2}} + \frac{g}{2} \left[\overline{h}_j (B_{j+\frac{1}{2}} - B_{j-\frac{1}{2}}) + \frac{n^2}{\overline{h}_j^{\,7/3}} |\bar{q}_j| \bar{q}_j \Delta x \right]$$

- Apply the minmod reconstruction procedure to $\{K_j, L_j\}$ and obtain the point values at the cell interfaces:

$$q_j^{\mathrm{E,W}} = q_j \pm \frac{\Delta x}{2}(q_x)_j, \quad K_j^{\mathrm{E,W}} = K_j \pm \frac{\Delta x}{2}(K_x)_j$$

Well-Balanced Algorithm

- Compute point values $h_j^{\mathrm{E,W}}$ by solving the nonlinear algebraic equations

$$\varphi(h) := \frac{(q_j^{\mathrm{E,W}})^2}{h} + \frac{g}{2}h^2 + R_{j\pm\frac{1}{2}} - K_j^{\mathrm{E,W}} = 0,$$

which does not have any positive solutions unless

$$\left(q_j^{\mathrm{E,W}}\right)^4 \le \frac{8(K_j^{\mathrm{E,W}} - R_{j\pm\frac{1}{2}})^3}{27g}$$

Consider h_j^{E}

Well-Balanced Algorithm

- If the inequality is not satisfied, we reconstruct $w = \bar{h} + B$ and set

$$h_j^{\mathrm{E}} = w_j^{\mathrm{E}} - B_{j+\frac{1}{2}} \qquad (*)$$

- If the inequality is satisfied, then
  - If $q_j^{\mathrm{E}} = 0$, then $h_j^{\mathrm{E}} = \sqrt{\dfrac{2\big(K_j^{\mathrm{E}} - R_{j+\frac{1}{2}}\big)}{g}}$
  - If $q_j^{\mathrm{E}} \neq 0$, then

$$h_j^{\mathrm{E}} = 2\sqrt{P}\cos\left(\frac{1}{3}\big[\Theta + 2\pi k\big]\right), \quad k = 0, 1, 2,$$

where

$$P := \frac{2\big(K_j^{\mathrm{E}} - R_{j+\frac{1}{2}}\big)}{3g} \quad \text{and} \quad \Theta := \arccos\left(-\frac{\big(q_j^{\mathrm{E}}\big)^2}{gP^{3/2}}\right)$$

Only two roots are positive (subsonic and supersonic cases). We single out the physically relevant solution by choosing a root that is closer to the corresponding value of h_j^{E} given in $(*)$

Well-Balanced Algorithm

- Update the cell averages in time:

$$\frac{d}{dt}\bar{h}_j = \frac{a^+_{j+\frac{1}{2}} q^{\mathrm{E}}_j - a^-_{j+\frac{1}{2}} q^{\mathrm{W}}_{j+1}}{a^+_{j+\frac{1}{2}} - a^-_{j+\frac{1}{2}}} + \frac{a^+_{j+\frac{1}{2}} a^-_{j+\frac{1}{2}}}{a^+_{j+\frac{1}{2}} - a^-_{j+\frac{1}{2}}} (h^{\mathrm{W}}_{j+1} - h^{\mathrm{E}}_j)$$

$$\frac{d}{dt}\bar{q}_j = \frac{a^+_{j+\frac{1}{2}} K^{\mathrm{E}}_j - a^-_{j+\frac{1}{2}} K^{\mathrm{W}}_{j+1}}{a^+_{j+\frac{1}{2}} - a^-_{j+\frac{1}{2}}} + \frac{a^+_{j+\frac{1}{2}} a^-_{j+\frac{1}{2}}}{a^+_{j+\frac{1}{2}} - a^-_{j+\frac{1}{2}}} (q^{\mathrm{W}}_{j+1} - q^{\mathrm{E}}_j)$$

with

$$a^+_{j+\frac{1}{2}} = \max\left\{u^{\mathrm{W}}_{j+1} + \sqrt{gh^{\mathrm{W}}_{j+1}}, u^{\mathrm{E}}_j + \sqrt{gh^{\mathrm{E}}_j}, 0\right\}$$

$$a^-_{j+\frac{1}{2}} = \min\left\{u^{\mathrm{W}}_{j+1} + \sqrt{gh^{\mathrm{W}}_{j+1}}, u^{\mathrm{E}}_j + \sqrt{gh^{\mathrm{E}}_j}, 0\right\}$$

Numerical Tests

Example 1 – Accuracy Test, No Friction

- Initial data and the bottom topography: function are

$$h(x,0)=5+e^{\cos(2\pi x)}, \quad q(x,0)=\sin(\cos(2\pi x)), \quad B(x)=\sin^2(\pi x)$$

- 1-periodic boundary conditions are imposed on $[0,1]$
- Reference solution is computed on a very fine mesh with 51200 uniform grid cells, time is $t=0.1$.

Number of grid cells	h		q	
	L^1-error	Rate	L^1-error	Rate
50	1.51e-03	–	1.21e-00	–
100	3.06e-04	2.30	2.26e-01	2.42
200	6.68e-05	2.20	4.90e-02	2.21
400	1.54e-05	2.12	1.17e-02	2.06
800	3.76e-06	2.04	2.95e-03	1.99
1600	9.29e-07	2.02	7.34e-04	2.01

Example 2 – Convergence to Steady States, no Friction

- Three sets of initial conditions:
  - Supercritical flow with

$$h(x,0)=2-B(x),\quad q(x,0)\equiv 0,$$
$$h(0,t)=2,\quad q(0,t)=24;$$

  - Subcritical flow with

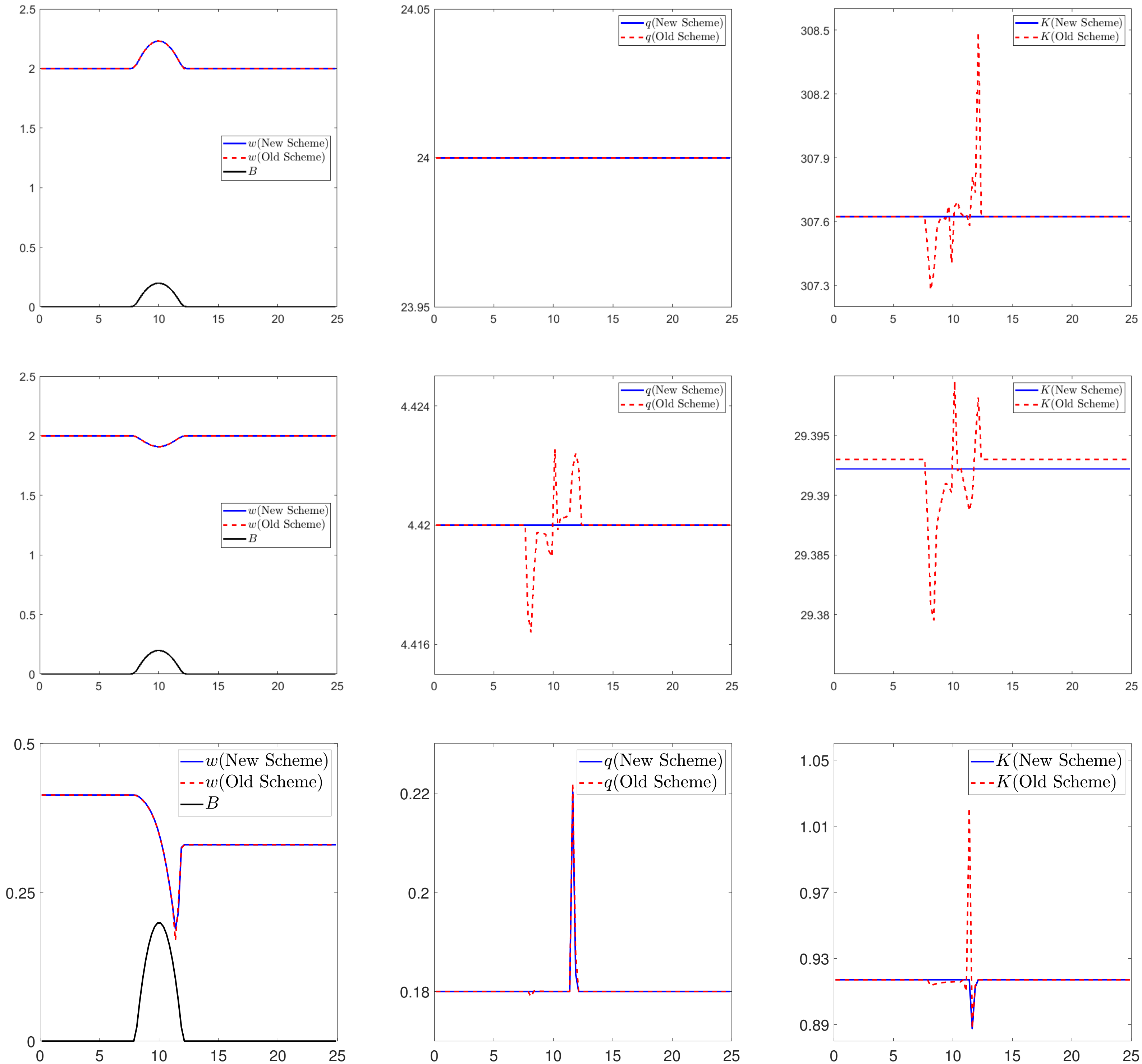
$$h(x,0)=2-B(x),\quad q(x,0)\equiv 0,$$
$$q(0,t)=4.42,\quad h(25,t)=2;$$

  - Transcritical flow without a shock with

$$h(x,0)=0.66-B(x),\quad q(x,0)\equiv 0,$$
$$q(0,t)=1.53,\quad h(25,t)=0.66.$$

- **Continuous** bottom topography:

$$B(x)=\begin{cases} 0.2-0.05(x-10)^2, & \text{if } 8\le x\le 12,\\ 0, & \text{otherwise.}\end{cases}$$


2.5
2
1.5
1
0.5
0
0 5 10 15 20 25
w(New Scheme)
w(Old Scheme)
B
24.05
24
23.95
0 5 10 15 20 25
q(New Scheme)
q(Old Scheme)
308.5
308.2
307.9
307.6
307.3
0 5 10 15 20 25
K(New Scheme)
K(Old Scheme)
2.5
2
1.5
1
0.5
0
0 5 10 15 20 25
w(New Scheme)
w(Old Scheme)
B
4.424
4.42
4.416
0 5 10 15 20 25
q(New Scheme)
q(Old Scheme)
29.395
29.39
29.385
29.38
0 5 10 15 20 25
K(New Scheme)
K(Old Scheme)
0.5
0.25
0
0 5 10 15 20 25
w(New Scheme)
w(Old Scheme)
B
0.22
0.2
0.18
0 5 10 15 20 25
q(New Scheme)
q(Old Scheme)
1.05
1.01
0.97
0.93
0.89
0 5 10 15 20 25
K(New Scheme)
K(Old Scheme)

Example 3 – Convergence to Steady States, with Friction ($n = 0.5$)

- The same three sets of initial conditions as in Example 2:
  - Supercritical flow with

$$h(x,0) = 2 - B(x), \quad q(x,0) \equiv 0,$$
$$h(0,t) = 2, \quad q(0,t) = 24;$$

  - Subcritical flow with

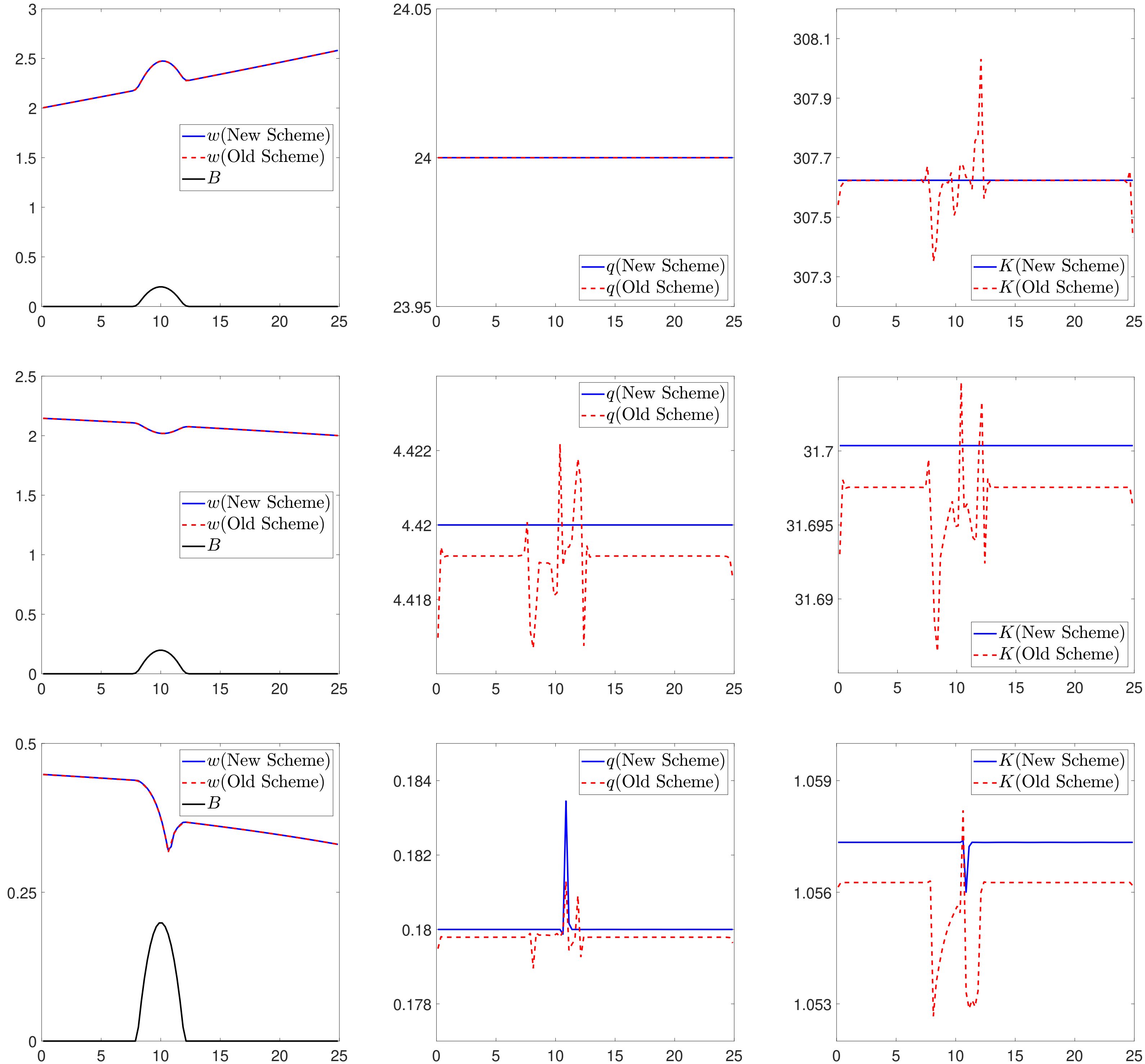
$$h(x,0) = 2 - B(x), \quad q(x,0) \equiv 0,$$
$$q(0,t) = 4.42, \quad h(25,t) = 2;$$

  - Transcritical flow without a shock with

$$h(x,0) = 0.66 - B(x), \quad q(x,0) \equiv 0,$$
$$q(0,t) = 1.53, \quad h(25,t) = 0.66.$$

- **Continuous** bottom topography:

$$B(x) = \begin{cases} 0.2 - 0.05(x-10)^2, & \text{if } 8 \le x \le 12, \\ 0, & \text{otherwise.} \end{cases}$$



w(New Scheme)
w(Old Scheme)
B
q(New Scheme)
q(Old Scheme)
K(New Scheme)
K(Old Scheme)

Example 4 – Convergence to Steady States, with Friction ($n = 0.5$)

- The same three sets of initial conditions as in Example 2:
  - Supercritical flow with

$$h(x,0) = 2 - B(x), \quad q(x,0) \equiv 0,$$
$$h(0,t) = 2, \quad q(0,t) = 24;$$

  - Subcritical flow with

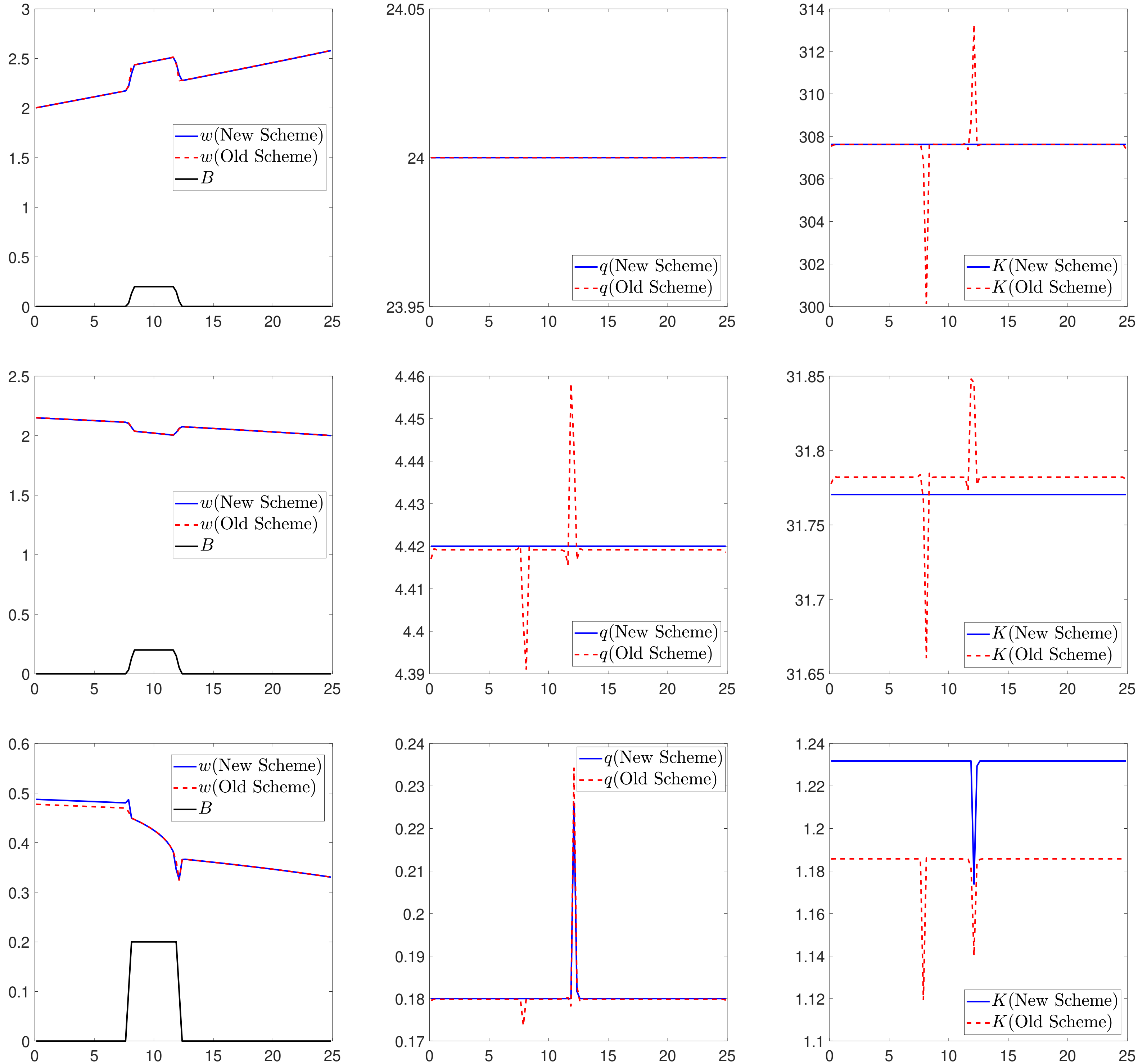
$$h(x,0) = 2 - B(x), \quad q(x,0) \equiv 0,$$
$$q(0,t) = 4.42, \quad h(25,t) = 2;$$

  - Transcritical flow without a shock with

$$h(x,0) = 0.66 - B(x), \quad q(x,0) \equiv 0,$$
$$q(0,t) = 1.53, \quad h(25,t) = 0.66.$$

- **Discontinuous** bottom topography:

$$B(x) = \begin{cases} 0.2, & \text{if } 8 \le x \le 12, \\ 0, & \text{otherwise.} \end{cases}$$


w(New Scheme)
w(Old Scheme)
B
q(New Scheme)
q(Old Scheme)
K(New Scheme)
K(Old Scheme)
w(New Scheme)
w(Old Scheme)
B
q(New Scheme)
q(Old Scheme)
K(New Scheme)
K(Old Scheme)
w(New Scheme)
w(Old Scheme)
B
q(New Scheme)
q(Old Scheme)
K(New Scheme)
K(Old Scheme)

Example 5 – Small Perturbations of Moving-Water Equilibria, with Friction ($n = 0.5$)

- Two sets of initial conditions:
  - Supercritical flow with

$$q(x,0) \equiv 24, \quad K(x,0) \equiv 307.624,$$

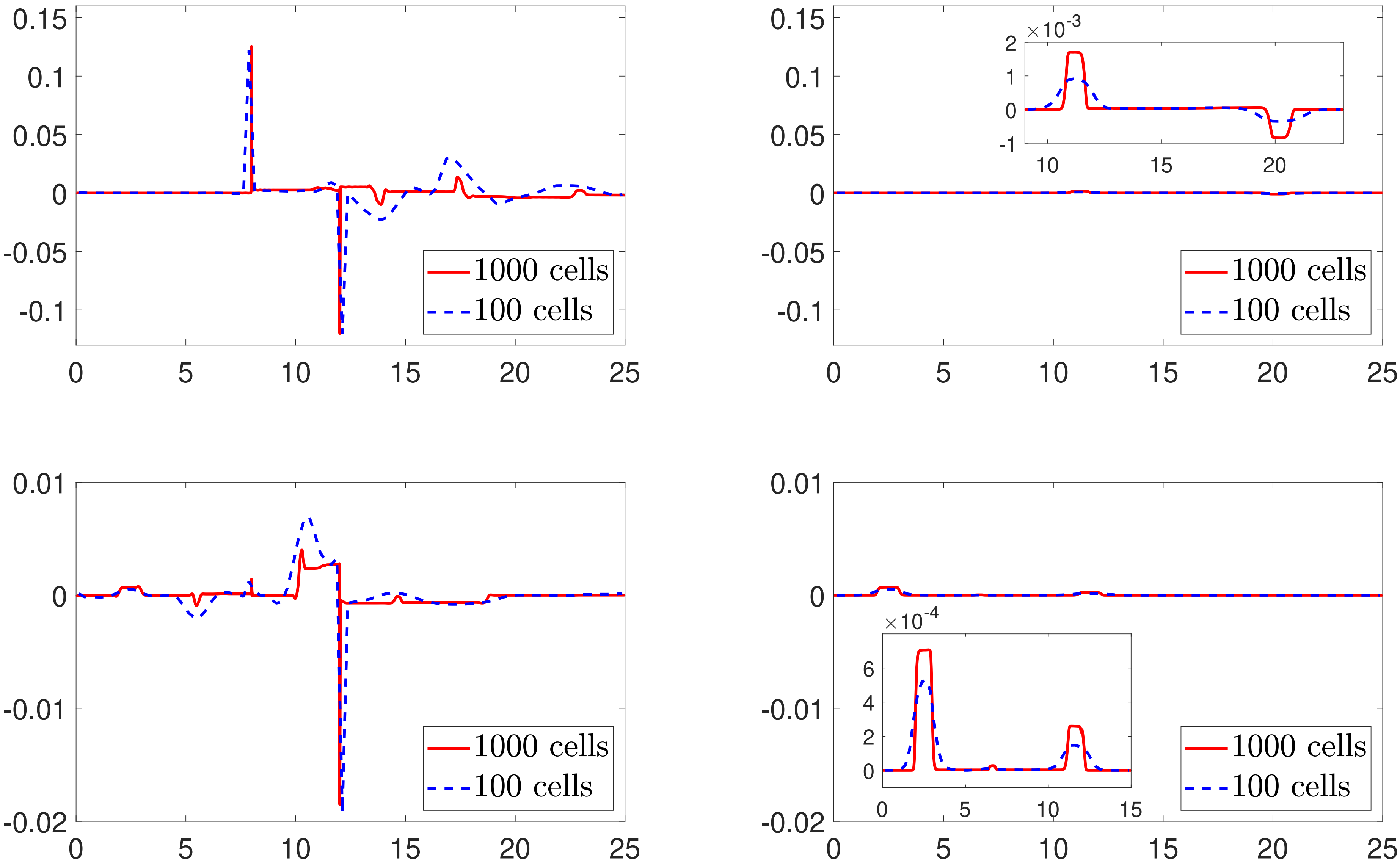
  - Subcritical flow with

$$q(x,0) \equiv 4.42, \quad K(x,0) \equiv 31.7705,$$

- **Discontinuous** bottom topography:

$$B(x) = \begin{cases} 0.2, & \text{if } 8 \le x \le 12, \\ 0, & \text{otherwise.} \end{cases}$$

We add 0.001 for $x \in [4.5, 5.5]$ to the corresponding water depth. We compute the solutions until the final time $t = 1$ using either 100 or 1000 uniform grid cells.


1000 cells
100 cells
×10^-3
×10^-4

Shallow Water System with Coriolis Force

$$\begin{cases} h_t + (hu)_x + (hv)_y = 0 \\ (hu)_t + \left(hu^2 + \dfrac{g}{2}h^2\right)_x + (huv)_y = -ghB_x + fhv \\ (hv)_t + (huv)_x + \left(hv^2 + \dfrac{g}{2}h^2\right)_x = -ghB_y - fhu \end{cases}$$

- h: water height
- u, v: fluid velocity
- B: bottom topography
- g: gravitational constant
- f: Coriolis parameter; $f \equiv 0 \implies$ Saint Venant system of shallow water.

Steady States

$$\begin{cases} h_t + (hu)_x + (hv)_y = 0 \\ (hu)_t + \left(hu^2 + \frac{g}{2}h^2\right)_x + (huv)_y = -ghB_x + fhv \\ (hv)_t + (huv)_x + \left(hv^2 + \frac{g}{2}h^2\right)_y = -ghB_y - fhu \end{cases}$$

- "Lake at rest": $u \equiv 0,\ v \equiv 0,\ h + B \equiv \text{Const}$
- Geostrophic equlibria ("jets in the rotational frame") are both stationary and constant along the streamlines:

$$u \equiv 0,\ v_y \equiv 0,\ h_y \equiv 0,\ B_y \equiv 0,\ K \equiv \text{Const}$$

$$v \equiv 0,\ u_x \equiv 0,\ h_x \equiv 0,\ B_x \equiv 0,\ L \equiv \text{Const}$$

Here,

$$K := g(h + B - V) \quad \text{and} \quad L := g(h + B + U)$$

are the potential energies defined through the primitives of the Coriolis force $(U, V)^T$:

$$V_x := \frac{f}{g}v \quad \text{and} \quad U_y := \frac{f}{g}u$$

2-D Well-Balanced Scheme

- Define

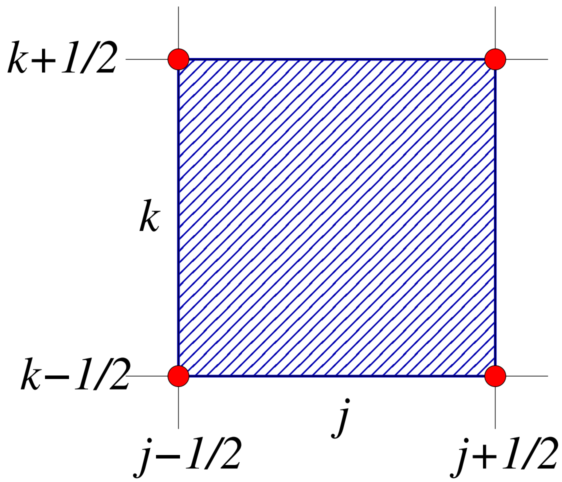
  conservative variables: $\boldsymbol{U} := (h, hu, hv)^T$

  equilibrium variables: $\boldsymbol{W} := (u, v, K, L)^T$

  fluxes in the x- and y-directions: $\boldsymbol{f}(\boldsymbol{U}, B)$ and $\boldsymbol{g}(\boldsymbol{U}, B)$

- Assume that at time t the cell averages are available

$$\overline{\boldsymbol{U}}_{j,k}(t) := \frac{1}{\Delta x \Delta y} \iint\limits_{C_{j,k}} \boldsymbol{U}(x, y, t)\, dxdy,$$



- Solve by the well-balanced scheme

$$\{\overline{\boldsymbol{U}}_{j,k}(t)\} \to \widetilde{\boldsymbol{U}}(\cdot, t) \to \left\{\mathbf{W}^{\mathrm{E,W,N,S}}_{j,k}(t)\right\} \to \left\{\boldsymbol{U}^{\mathrm{E,W,N,S}}_{j,k}(t)\right\}$$
$$\to \left\{\boldsymbol{\mathcal{F}}_{j+\frac{1}{2},k}(t), \boldsymbol{\mathcal{G}}_{j,k+\frac{1}{2}}(t)\right\} \to \{\overline{\boldsymbol{U}}_{j,k}(t + \Delta t)\}$$

Example — 2-D Stationary Vortex

[E. Audusse, R. Klein, D. D. Nguyen, and S. Vater, 2011]

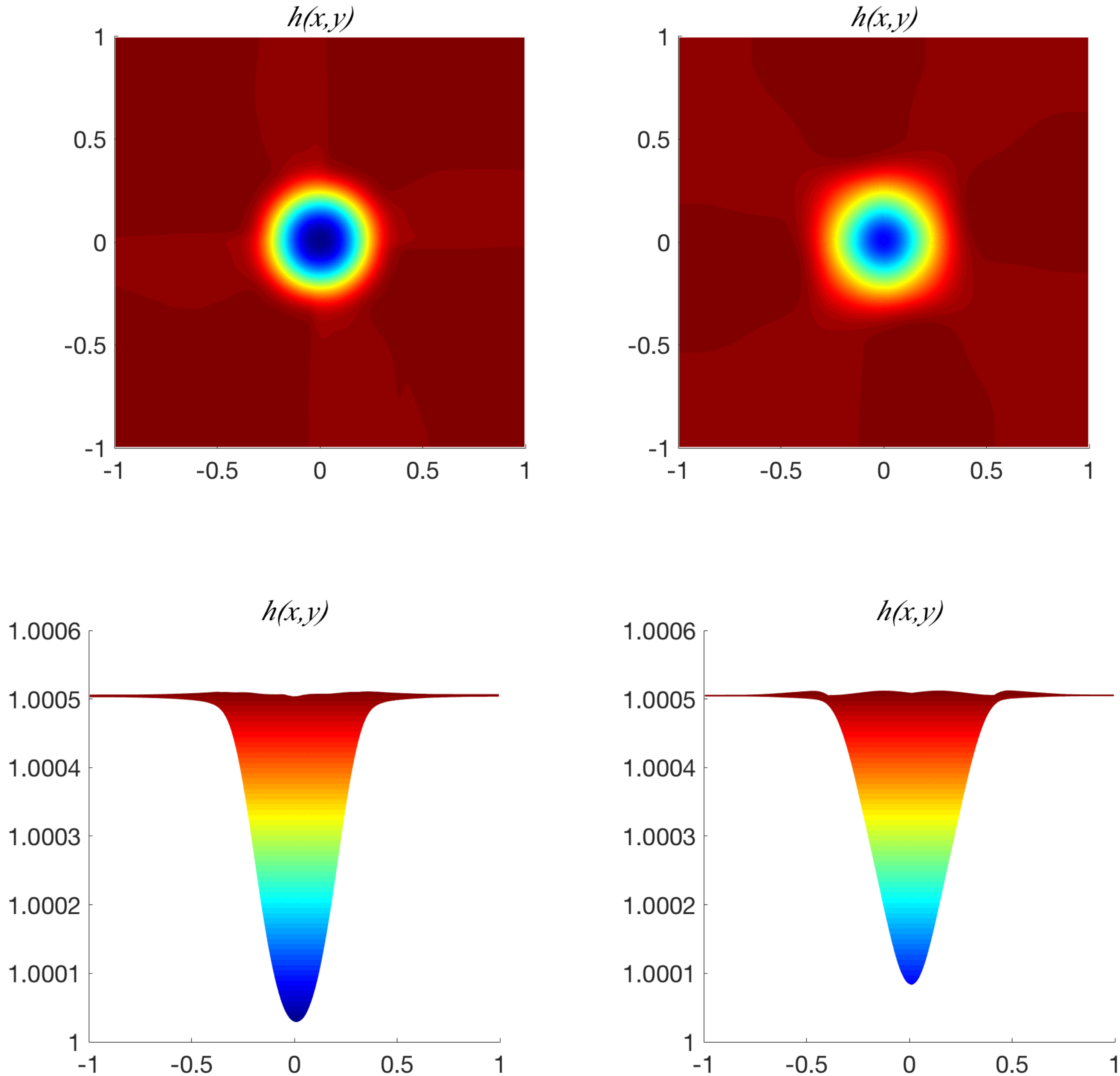
$$h(r,0) = 1+\varepsilon^2 \begin{cases} \frac{5}{2}(1+5\varepsilon^2)r^2 \\ \frac{1}{10}(1+5\varepsilon^2) + 2r - \frac{1}{2} - \frac{5}{2}r^2 + \varepsilon^2(4\ln(5r) + \frac{7}{2} - 20r + \frac{25}{2}r^2) \\ \frac{1}{5}(1 - 10\varepsilon + 4\varepsilon^2 \ln 2), \end{cases}$$

$$u(x,y,0) = -\varepsilon y \Upsilon(r), \quad v(x,y,0) = \varepsilon x \Upsilon(r), \quad \Upsilon(r) := \begin{cases} 5, & r < \frac{1}{5} \\ \frac{2}{r} - 5, & \frac{1}{5} \le r < \frac{2}{5} \\ 0, & r \ge \frac{2}{5}, \end{cases}$$

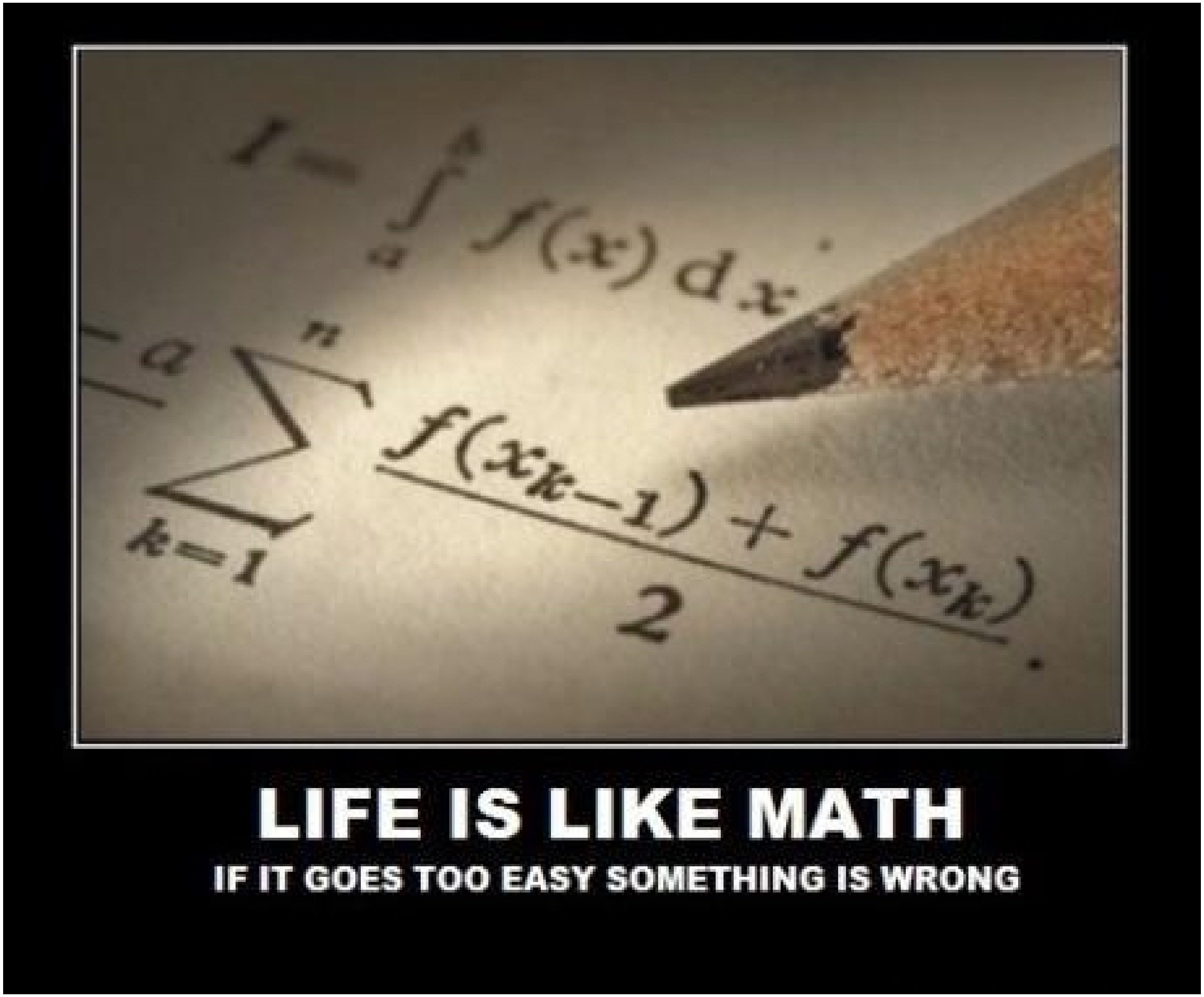
Domain: $[-1,1] \times [-1,1], \quad r := \sqrt{x^2+y^2}$

Boundary conditions: a zero-order extrapolation in both x- and y-directions

Parameters: $B \equiv 0, \quad f = 1/\varepsilon$ and $g = 1/\varepsilon^2$ with $\varepsilon = 0.05$



h(x,y)
h(x,y)
h(x,y)
h(x,y)


LIFE IS LIKE MATH
IF IT GOES TOO EASY SOMETHING IS WRONG

Asymptotic Perserving Methods

Explicit Discretization

Eigenvalues of the flux Jacobian:

$$\left\{u \pm \frac{1}{\varepsilon}\sqrt{h},\, u\right\} \quad \text{and} \quad \left\{v \pm \frac{1}{\varepsilon}\sqrt{h},\, v\right\}$$

This leads to the CFL condition

$$\Delta t_{\mathrm{expl}} \leq \nu \cdot \min\left(\frac{\Delta x}{\max\limits_{u,h}\left\{|u| + \frac{1}{\varepsilon}\sqrt{h}\right\}}, \frac{\Delta y}{\max\limits_{v,h}\left\{|v| + \frac{1}{\varepsilon}\sqrt{h}\right\}}\right) = \mathcal{O}(\varepsilon \Delta_{\min}).$$

where $\Delta_{\min} := \min(\Delta x, \Delta y)$

- $0 < \nu \leq 1$ is the CFL number
- Numerical diffusion: $\mathcal{O}(\lambda_{max} \Delta x) = \mathcal{O}(\varepsilon^{-1} \Delta x)$.
- We must choose $\Delta x \approx \varepsilon$ to control numerical diffusion and the stability condition becomes

$$\Delta t = \mathcal{O}(\varepsilon^2)$$

Low Froude Number Flows

Low Froude number regime ($0 < \varepsilon \ll 1$) $\Longrightarrow$ very large propagation speeds

Explicit methods:

- very restrictive time and space dicretization steps, typically proportional to ε due to the CFL condition;
- too computationally expensive and typically impractical.

Implicit schemes:

- uniformly stable for $0 < \varepsilon < 1$;
- may be inconsistent with the limit problem;
- may provide a wrong solution in the zero Froude number limit.

Goal: to design robust numerical algorithms, whose accuracy and efficiency is independent of ε

Some Refernces

- Harlow, Welch; 1965
- Chorin; 1967
- Harlow, Amsden; 1971
- Klainerman, Majda; 1981
- Turkel; 1987
- Abarbanel, Duth, Gottlieb; 1989
- Gustafsson, Stoor; 1991
- Klein; 1995
- Colella, Pao; 1999
- Guillard, Viozat; 1999
- Guillard, Murrone; 2004
- Kadioglu, Sussman, Osher, Wright, Kang; 2005

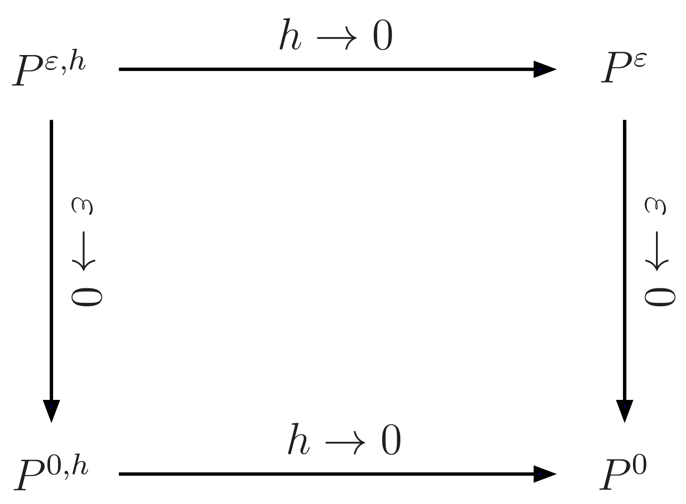
Asymptotic-Preserving (AP) Methods

Introduced in [Klar; 1998, Jin; 1999], see also [Jin, Levermore; 1991],

[Golse, Jin, Levermore; 1999].

Idea:

- asymptotic passage from one model to another should be preserved at the discrete level;
- for a fixed mesh size and time step, AP method should automatically transform into a stable discretization of the limitting model as $\varepsilon \to 0$.



AP Methods – References

[Degond, Jin, Liu; 2007]

[Degond, Hua, Navoret; 2011]

[Degond and M. Tang; 2011]

[Berthon, Turpault; 2011]

[Cordier, Degond, Kumbaro; 2012]

[Haack, Jin, Liu; 2012]

[Arun, Noelle, Lukáčová-Medvid'ová, Munz; 2014]

[Miczek, Roepke, Edelmann; 2015]

[Bispen, Lukáčová-Medvid'ová, Yelash; 2017]

[Feireisl, Klingenberg, Markfelder; preprint 2017]

Though the existing AP schemes work perfectly well for many simpler models, their applicability to more complicated systems is rather limited: They works very well for large ($\varepsilon \sim 1$) and intermediate ($\varepsilon \sim 10^{-1}$) values of ε, but may become inefficient for smaller ε numbers.

Theorem. *A new hyperbolic flux splitting method coupled with the described fully discrete scheme, which is asymptotic preserving in the sense that it provides a consistent and stable discretization of the limiting system as the Froude number* $\varepsilon \to 0$.

Joint work with Alexander Kurganov and Xin Liu

Example — 2-D Stationary Vortex

[E. Audusse, R. Klein, D. D. Nguyen, and S. Vater, 2011]

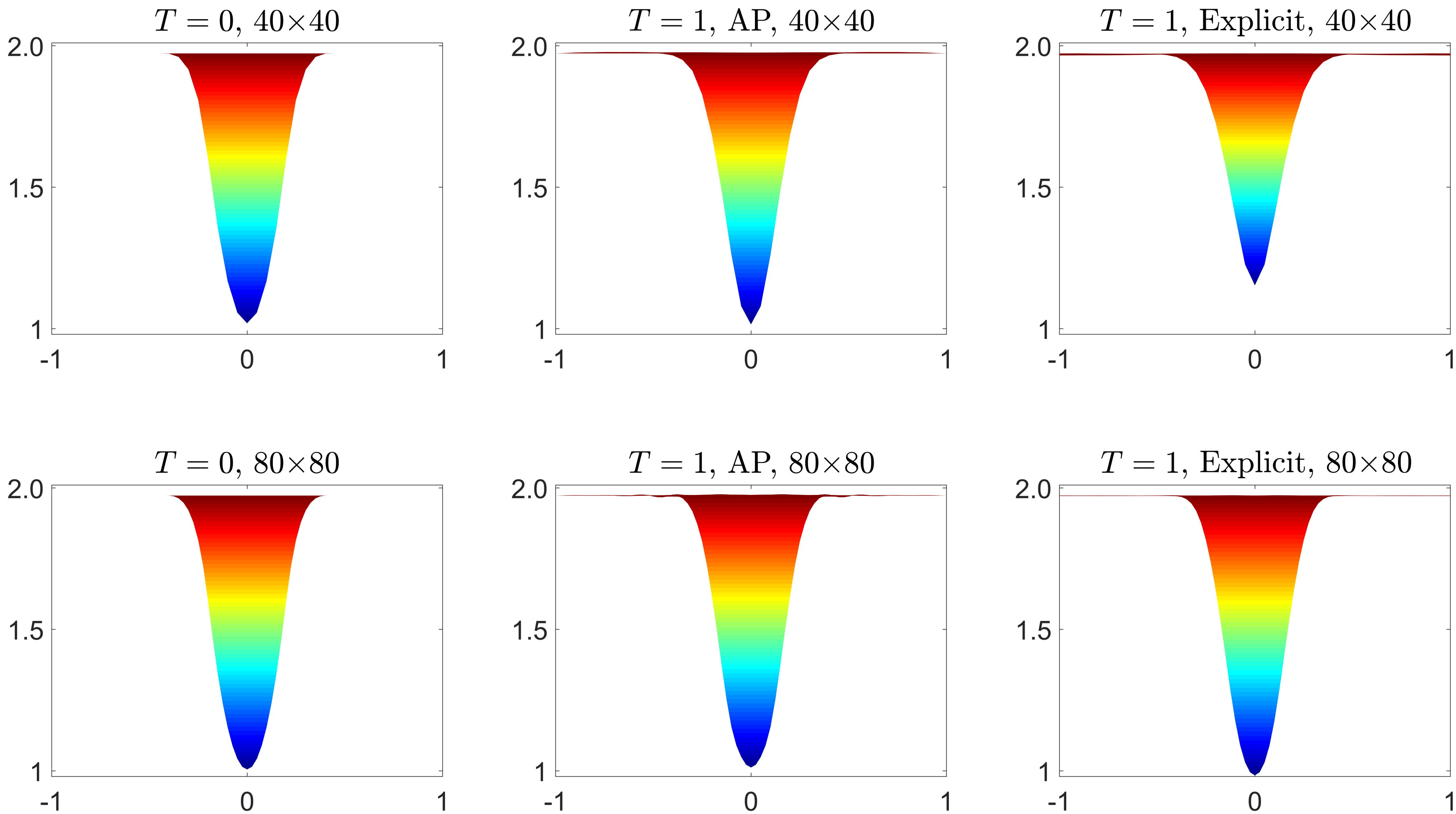
$$h(r,0) = 1+\varepsilon^2 \begin{cases} \frac{5}{2}(1+5\varepsilon^2)r^2 \\ \frac{1}{10}(1+5\varepsilon^2) + 2r - \frac{1}{2} - \frac{5}{2}r^2 + \varepsilon^2(4\ln(5r) + \frac{7}{2} - 20r + \frac{25}{2}r^2) \\ \frac{1}{5}(1 - 10\varepsilon + 4\varepsilon^2 \ln 2), \end{cases}$$

$$u(x,y,0) = -\varepsilon y \Upsilon(r), \quad v(x,y,0) = \varepsilon x \Upsilon(r), \quad \Upsilon(r) := \begin{cases} 5, & r < \frac{1}{5} \\ \frac{2}{r} - 5, & \frac{1}{5} \le r < \frac{2}{5} \\ 0, & r \ge \frac{2}{5}, \end{cases}$$

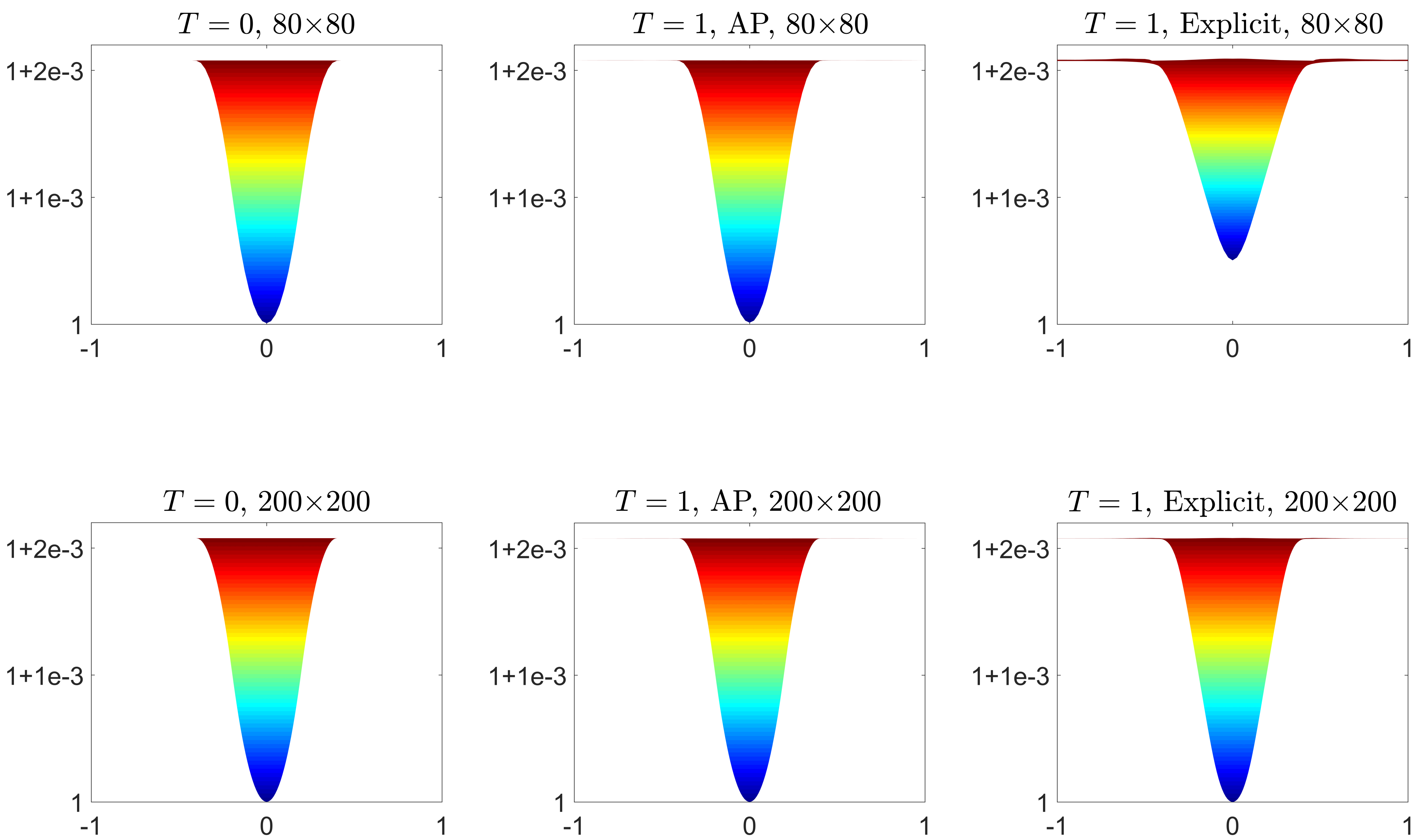
Domain: $[-1,1] \times [-1,1], \quad r := \sqrt{x^2+y^2}$

Boundary conditions: a zero-order extrapolation in both x- and y-directions

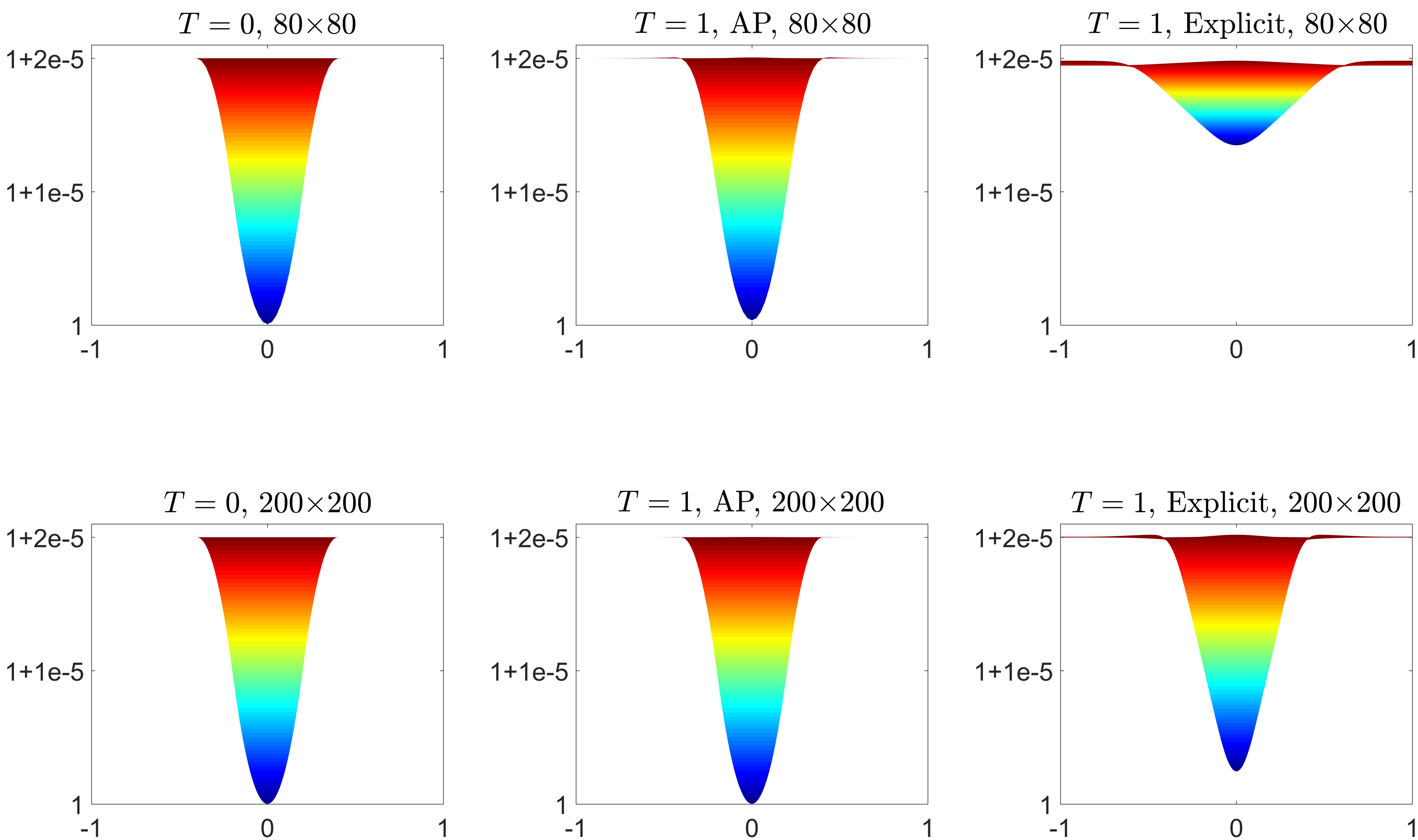
Comparison of non-AP and AP methods, $\varepsilon = 1$



Comparison of non-AP and AP methods, $\varepsilon = 0.1$



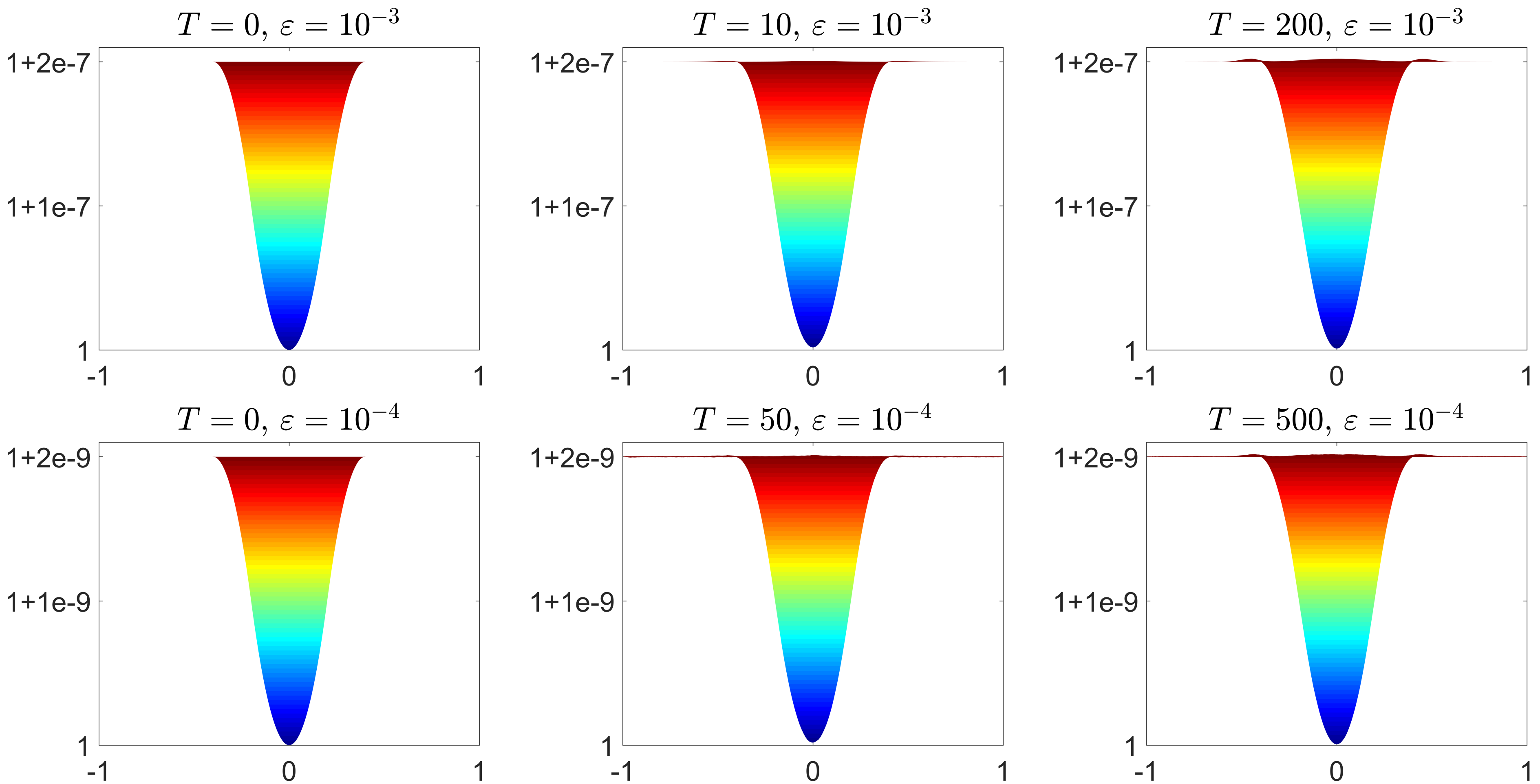
Comparison of non-AP and AP methods, $\varepsilon = 0.01$



Comparison of non-AP and AP methods, CPU times

	$\varepsilon = 1$		$\varepsilon = 0.1$		$\varepsilon = 0.01$	
Grid	AP	Explicit	AP	Explicit	AP	Explicit
40×40	0.18 s	0.16 s	0.06 s	1.25 s	0.03 s	10.53 s
80×80	1.57 s	1.32 s	0.29 s	4.73 s	0.18 s	47.0 s
200×200	24.11 s	21.36 s	5.36 s	163.36 s	3.37 s	804.15 s

Smaller values: $\varepsilon = 10^{-3}$ and $\varepsilon = 10^{-4}$



Smaller times: 200×200, larger times: 500×500

THANK YOU!