HELICITY OF SEMICLASSICAL CHIRAL PARTICLES

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Abstract : The twisted Lorentz symmetry of semiclassical chiral fermions with "enslved spin" could be recovered by embedding them into Souriau's relativistic massless spinning particle. The latter has "unchained" spin. Helicity, defined as the projection of spin onto the momentum direction, is automatically conserved for the chiral fermion, whereas non-minimal spin-field coupling endows the massless spin-extended particle with an effective mass that breaks helicity conservation. An explicit example is presented.

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Based on :

[1] C. Duval and P. A. Horvathy, *"Chiral fermions as classical massless spinning particles,"* Phys. Rev. **D91** 045013 (2015) arXiv:1406.0718 [hep-th].

[2] C. Duval, M. Elbistan, P. A. Horvathy and P.-M. Zhang, *"Wigner-Souriau translations and Lorentz symmetry of chiral fermions,"* Phys. Lett. **B 742** (2015) 322, arXiv:1411.6541 [hep-th].

[3] M. Elbistan and P. A. Horvathy, *"Anomalous properties of spin-extended chiral fermions,"* Phys. Lett. **B749** (2015) 502. arXiv:1506.05008 [hep-th].

[4] C. Duval, M. Elbistan, P. A. Horvathy and P.-M. Zhang, *"Helicity of spin-extended chiral particles,"* Phys. Lett. **A 380** 1677 (2016) arXiv:1508.02188 [hep-th].

Helicity

The Feynman Lectures on Physics, III. Sec. 17-4

† We have tried to find at least a proof that the component of angular momentum along the direction of motion must for a zero mass particle be an integral multiple of $\hbar/2$—and not something like $\hbar/3$. Even using all sorts of properties of the Lorentz transformation and what not, we failed. Maybe it's not true. We'll have to talk about it with Prof. Wigner, who knows all about such things.

17–11

THM : Ang mom $\mathbf{J} = \boldsymbol{x} \times \boldsymbol{p} + \boldsymbol{s}$ Helicity :

$$\chi = \mathbf{J} \cdot \widehat{\boldsymbol{p}} = \boldsymbol{s} \cdot \widehat{\boldsymbol{p}}$$

conserved for light & for chiral fermions minimally coupled a gauge/gravitational field,

However :

THM : may not be conserved for spin-extended chiral fermions non-minimally coupled to a gauge field.

Clue: non-minimal coupling endows the particle with an effective mass

$$P_\mu P^\mu = -(eg/2)\, S.F$$

g gyromagnetic ratio; $S.F = S_{\mu\nu}F^{\mu\nu}$ spin-field coupling.

SEMICLASSICAL FERMIONS
(c-model)

Stephanov & Yin PRL 2012 (semi)classical model derived from Weyl Hamiltonian $H = \boldsymbol{\sigma} \cdot \boldsymbol{p}$ $\rightsquigarrow$ spin-1/2 system

$$S = \int \Big(\big(\boldsymbol{p} + e\boldsymbol{A} \big) \cdot \frac{d\boldsymbol{x}}{dt} - \big(|\boldsymbol{p}| + e\phi \big) - \boxed{\boldsymbol{a} \cdot \frac{d\boldsymbol{p}}{dt}} \Big) dt \qquad (1)$$

$\boldsymbol{a}(\boldsymbol{p})$ "momentum-dependent vector potential" for "Berry monopole" in $\boldsymbol{p}$-space

$$\boldsymbol{\nabla}_{\boldsymbol{p}} \times \boldsymbol{a} = \boldsymbol{\Theta} \equiv \frac{\widehat{\boldsymbol{p}}}{2|\boldsymbol{p}|^2}, \qquad \widehat{\boldsymbol{p}} = \boldsymbol{p}/|\boldsymbol{p}|. \qquad (2)$$

Spin "enslaved" : angular momentum

$$\mathbf{J} = \boldsymbol{x} \times \boldsymbol{p} + \boldsymbol{s}, \qquad \boldsymbol{s} = \pm\tfrac{1}{2}\,\widehat{\boldsymbol{p}}. \qquad (3)$$

Helicity = projection of spin onto momentum :

$$\chi = \widehat{\boldsymbol{p}} \cdot \boldsymbol{s} = \pm\tfrac{1}{2} \qquad (4)$$

automatically conserved : built into model.

Chiral model has **No** *manifest Lorentz symmetry*.

However, J. Y. Chen et al. PRL **113** (2014) 182302 found twisted Lorenz symmetry

$$
\begin{aligned}
\delta \boldsymbol{x} &= \boldsymbol{\beta}\, t + \boxed{\boldsymbol{\beta} \times \frac{\widehat{\boldsymbol{p}}}{2|\boldsymbol{p}|}}, \qquad (5)\\
\delta \boldsymbol{p} &= |\boldsymbol{p}|\, \boldsymbol{\beta},\\
\delta t &= \boldsymbol{\beta} \cdot \boldsymbol{x}.
\end{aligned}
$$

DH [1]: (5) derived from symmetry of massless spinning particle Souriau 1970.

Henceforce only consider Souriau's S-model, constructed by symmetry (orbit method).

Souriau's mechanics

Souriau 1970: Phase space variables $\boldsymbol{x}$, $\boldsymbol{p}$ denoted ξ^α. Particle described by phase space Lagrangian

$$\mathcal{L}(\xi,\dot{\xi}) = a_\alpha(\xi)\dot{\xi}^\alpha - h(\xi). \tag{6}$$

Euler-Lagrange var. eqns $\frac{d}{dt}\Big(\frac{\partial\mathcal{L}}{\partial\dot{\xi}^\alpha}\Big) - \frac{\partial\mathcal{L}}{\partial\xi^\alpha} = 0$:

$$\omega_{\alpha\beta}\dot{\xi}^\beta = -\partial_\alpha h\,, \quad \omega_{\alpha\beta} = \partial_\alpha a_\beta - \partial_\beta a_\alpha. \tag{7}$$

Extend 6d phase space into 7d evolution space V^7 described by $(\boldsymbol{x},\boldsymbol{p},t)$ and unify symplectic form $\omega = \frac{1}{2}\omega_{\alpha\beta}d\xi^\alpha \wedge d\xi^\beta$ with Hamiltonian

$$\sigma = \omega - dh \wedge dt. \tag{8}$$

Eqns of motion (7) become

$$\boxed{\sigma\Big((\dot{\boldsymbol{x}},\dot{\boldsymbol{p}},\dot{t}),\ \ \cdot\ \ \Big) = 0} \tag{9}$$

Tangent to motion in evolution space belongs to kernel $\ker\sigma$*.*

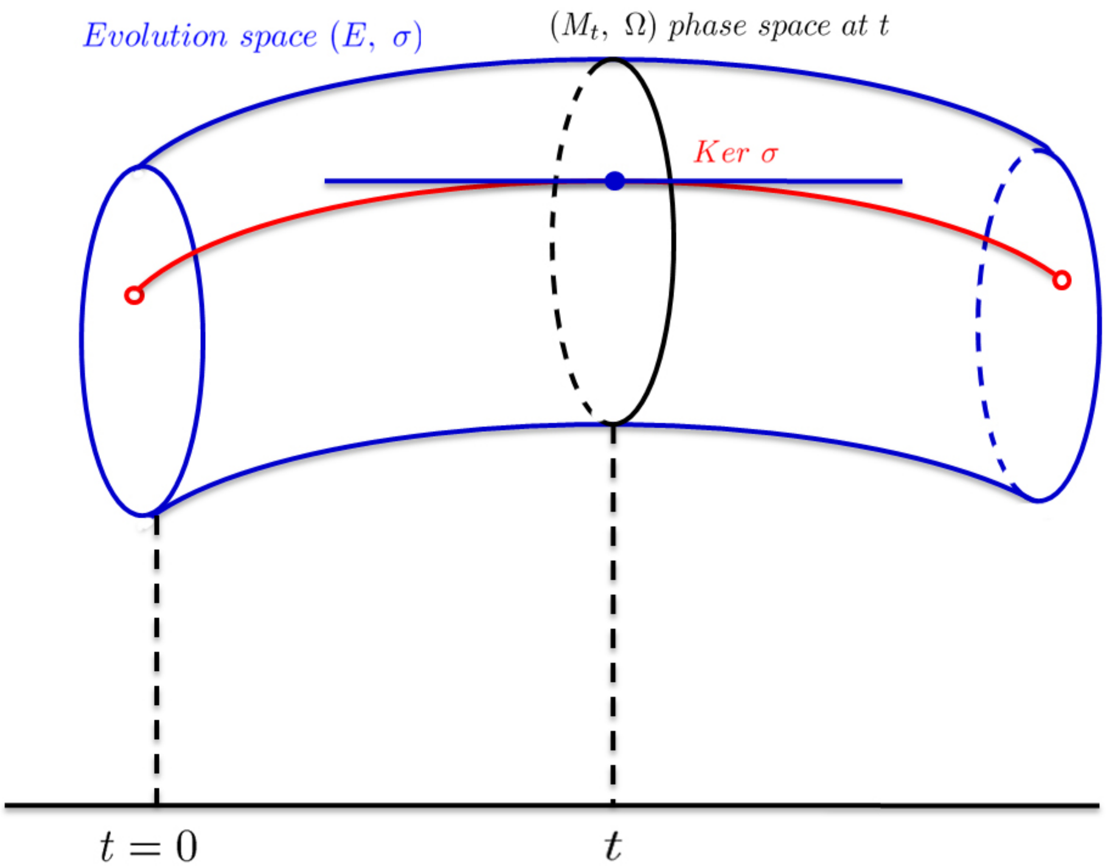
When matrix $[\omega_{\alpha\beta}]$ regular $\Rightarrow$ can be inverted $\Rightarrow$ (7) equivalent to Hamilton's eqns.

Ex. : for canonical symplectic form

$$\left[\omega_{\alpha\beta}\right]=\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix} \rightsquigarrow \left[\omega^{\alpha\beta}\right]=\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix} \quad (10)$$

$$\dot{\mathbf{q}}=\frac{\partial H}{\partial \boldsymbol{p}}, \qquad \dot{\boldsymbol{p}}=-\frac{\partial H}{\partial \mathbf{q}}. \quad (11)$$

Souriau : Classical system described by closed 2-form σ of const rank on evolution space V. Motion $\sim \ker\sigma$.



Framework viewed as Hamiltonian & Lagrangian formalism at same time.

Souriau's relativistic classical massless spinning model

S-model described by 9-dim evolution space V^9, labeled with Minkowski 4-vectors $R = (R^\mu)$ (position), $P = (P^\mu)$ (momentum) and spin tensor $S = (S_{\mu\nu}) \in \mathfrak{o}(3,1)$, subject to constraints

$$S_{\mu\nu}P^\nu = 0 \qquad (12a)$$

$$\tfrac{1}{2}S_{\mu\nu}S^{\mu\nu} = s^2 \qquad (12b)$$

$$P_\mu P^\mu = \boxed{-(eg/2)\,S.F} \qquad (12c)$$

(12c) spin-field-coupling ⇝ effective mass.

1. $g = 0$ minimal coupling; remains massless

2. $g \neq 0$ non-minimal coupling. Becomes massive;

3. $g = 2$ normal coupling. Consistent with Dirac eqn.

N.B. Künzle, Souriau, Duval 1970-74 : (12c) generalized ⇝ *any* fct. $f(S \cdot F)$ yields consistent system.

Eqns (12) derive from Souriau's two-form obtained by First Principles [group-theoretically constructed free model+geometrically-defined coupling+ mass constraint] :

$$\sigma = \underbrace{-dP_\mu \wedge dR^\mu - \frac{1}{2s^2}\, dS^\mu_\lambda \wedge S^\lambda_\rho\, dS^\rho_\mu}_{free+(12c)} \qquad (13)$$
$$+ \underbrace{\tfrac{1}{2}eF_{\mu\nu}\, dR^\mu \wedge dR^\nu}_{min.\, coupling}.$$

In Lorentz frame

$$R = (\mathbf{r}, t),\ P = (\boldsymbol{p}, \mathcal{E}),\ S_{ij} = \epsilon_{ijk} s_k,\ S_{i4} = \Big(\frac{\boldsymbol{p}}{\mathcal{E}} \times \boldsymbol{s}\Big)_i$$

where $\boldsymbol{s} = (s_i)$ spin vector.

Dispersion relation

$$\mathcal{E} = \sqrt{|\boldsymbol{p}|^2 - (eg/2)S \cdot F} \qquad (14)$$
$$= \sqrt{|\boldsymbol{p}|^2 - 2(eg/2)\boldsymbol{s} \cdot \Big(\boldsymbol{B} - \frac{\boldsymbol{p}}{\mathcal{E}} \times \boldsymbol{E}\Big)}\,.$$

NB: (14) follows from posited mass formula (12c). $\mathcal{E} = |\boldsymbol{p}|$ when $(eg/2)S \cdot F = 0$. Can *not* be changed arbitrarily.

Eqns of motion

$$
\begin{aligned}
\dot{\mathrm{r}} &= \frac{3g}{2(g+1)}\boldsymbol{p} - \left(\frac{g-2}{g+1}\right)\left[\frac{\boldsymbol{s}\cdot\boldsymbol{p}}{S\cdot F}\left(\boldsymbol{B} - \frac{\boldsymbol{p}}{\mathcal{E}}\times\boldsymbol{E}\right) - \frac{eg}{2}\boldsymbol{E}\times\frac{\boldsymbol{s}}{\mathcal{E}}\right] \\
&\quad - \frac{g}{2(g+1)S\cdot F}\left(\boldsymbol{s}\times(S\cdot\boldsymbol{D}F) - \frac{\boldsymbol{p}}{\mathcal{E}}\times\boldsymbol{s}\,(S\cdot D_t F)\right), \\
\dot{t} &= \frac{g}{2(g+1)\mathcal{E}}\left(3|\boldsymbol{p}|^2 - (g+1)eS\cdot F\right) \\
&\quad - \left(\frac{g-2}{g+1}\right)\tfrac{1}{\mathcal{E}\,S\cdot F}\,(\boldsymbol{p}\cdot\boldsymbol{B})(\boldsymbol{s}\cdot\boldsymbol{p}) + \frac{eg(g-2)}{2(g+1)\mathcal{E}^2}\boldsymbol{s}\cdot(\boldsymbol{p}\times\boldsymbol{E}) \\
&\quad - \frac{g}{(g+1)}\frac{1}{\mathcal{E}\,S\cdot F}(\boldsymbol{p}\times\boldsymbol{s})\cdot(S\cdot\boldsymbol{D}F), \\
\dot{\boldsymbol{p}} &= e\left(\boldsymbol{E}\,\dot{t} + \dot{\mathrm{r}}\times\boldsymbol{B}\right) + \frac{eg}{4}S\cdot\boldsymbol{D}F, \\
\dot{\boldsymbol{s}} &= \boldsymbol{p}\times\dot{\mathrm{r}} + \frac{eg}{2}\left(\left(\frac{\boldsymbol{p}}{\mathcal{E}}\times\boldsymbol{s}\right)\times\boldsymbol{E} + \boldsymbol{s}\times\boldsymbol{B}\right),
\end{aligned}
\tag{15}
$$

where introduced shorthands

$$
S\cdot D_j F = 2\boldsymbol{s}\cdot\left(\partial_j\boldsymbol{B} - \frac{\boldsymbol{p}}{\mathcal{E}}\times\partial_j\boldsymbol{E}\right), \tag{16}
$$

$$
S\cdot D_t F = 2\boldsymbol{s}\cdot\left(\partial_t\boldsymbol{B} - \frac{\boldsymbol{p}}{\mathcal{E}}\times\partial_t\boldsymbol{E}\right). \tag{17}
$$

N.B. : $\dfrac{d\mathbf{r}}{dt} = \dfrac{\dot{\mathbf{r}}}{\dot{t}}\ \ldots$

Zero rest-mass analog of Bargmann-Michel-Telegdi (BMT) eqns. for massive relativistic particles. To get insight: hence we limit ourselves to const. ext fields, $F = (\boldsymbol{E}, \boldsymbol{B})$.

For minimal coupling $g = 0$: $\mathcal{E} = |\boldsymbol{p}|$, eqns simplify :

$$(\widehat{\boldsymbol{p}} \cdot \boldsymbol{B})\frac{d\mathbf{r}}{dt} = \boldsymbol{B} - \widehat{\boldsymbol{p}} \times \boldsymbol{E}, \tag{18a}$$

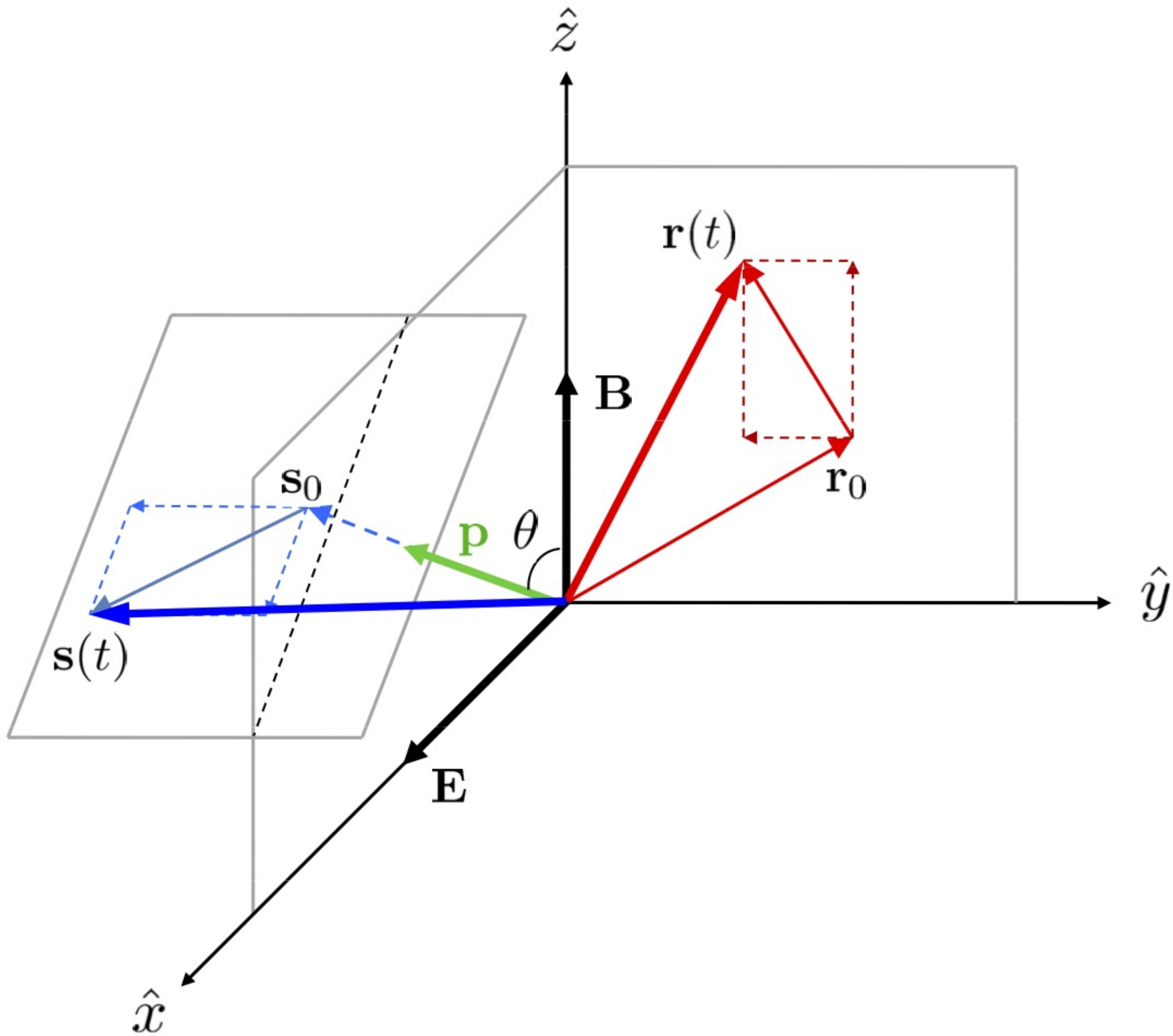
$$(\widehat{\boldsymbol{p}} \cdot \boldsymbol{B})\frac{d\boldsymbol{p}}{dt} = e(\boldsymbol{E} \cdot \boldsymbol{B})\widehat{\boldsymbol{p}}, \tag{18b}$$

$$(\widehat{\boldsymbol{p}} \cdot \boldsymbol{B})\frac{d\boldsymbol{s}}{dt} = \boldsymbol{p} \times \boldsymbol{B} - \boldsymbol{p} \times (\widehat{\boldsymbol{p}} \times \boldsymbol{E}). \tag{18c}$$

N.B. : (18a) "purely anomalous": *no* $\widehat{\boldsymbol{p}}$ term on rhs. Analyzed in [1].

Put, e.g., particle into crossed const. electric and magnetic fields ($\sim$ Hall effect, $\boldsymbol{E} = E\,\widehat{\boldsymbol{x}}$, $\boldsymbol{B} = B\,\widehat{\boldsymbol{z}}$.

$\boldsymbol{p}$ const of motion, $\rightsquigarrow$ so is angle between $\boldsymbol{B}$ and $\boldsymbol{p}$ ($\neq \pi/2$ to have $\widehat{\boldsymbol{p}} \cdot \boldsymbol{B} \equiv \det \neq 0$).



Motion in Hall-type electric and magnetic fields. Initial position, $\mathbf{r}_0$, chosen in y-z plane, initial **momentum** *and* **spin** *are chosen parallel ("enslaved") and in x-z plane. Then,* **spatial motion**, *$\mathbf{r}(t)$, is combination of constant-velocity Hall drift perpendicularly to $\boldsymbol{E}$, and $\boldsymbol{B}$, combined with constant-velocity vertical drift.* **Momentum**, *$\boldsymbol{p}$, is conserved, but* **spin**, *$\boldsymbol{s}(t)$, moves in plane perpendicular to momentum.*

Helicity

For minimal value $\boxed{g = 0}$ helicity constraint,

$$\widehat{\boldsymbol{p}} \cdot \boldsymbol{s} = \chi^{\circ}\,|s|, \qquad \chi^{\circ} = \pm 1 \equiv \mathrm{sign}(\boldsymbol{s}) \tag{19}$$

cf. (4), holds true but spin enslavement is broken. Conservation

$$\frac{d}{dt}\big(\widehat{\boldsymbol{p}} \cdot \boldsymbol{s}\big) = 0 \tag{20}$$

can be checked using eqns of motion (18), i.e.,

$$(\widehat{\boldsymbol{p}} \cdot \boldsymbol{B})\frac{d\mathbf{r}}{dt} = \boldsymbol{B} - \widehat{\boldsymbol{p}} \times \boldsymbol{E},$$

$$(\widehat{\boldsymbol{p}} \cdot \boldsymbol{B})\frac{d\boldsymbol{p}}{dt} = e(\boldsymbol{E} \cdot \boldsymbol{B})\widehat{\boldsymbol{p}},$$

$$(\widehat{\boldsymbol{p}} \cdot \boldsymbol{B})\frac{d\boldsymbol{s}}{dt} = \boldsymbol{p} \times \boldsymbol{B} - \boldsymbol{p} \times (\widehat{\boldsymbol{p}} \times \boldsymbol{E}).$$

Helicity conservation follows also from geometric framework [1].

For non-minimal value $\boxed{g=2}$, eqns of motion simplify again,

$$\frac{d\mathbf{r}}{dt} = \frac{\boldsymbol{p}}{\mathcal{E}}\,, \tag{22a}$$

$$\frac{d\boldsymbol{p}}{dt} = e\left(\boldsymbol{E} + \frac{\boldsymbol{p}}{\mathcal{E}} \times \boldsymbol{B}\right), \tag{22b}$$

$$\frac{d\boldsymbol{s}}{dt} = \frac{e}{\mathcal{E}}\left(\left(\frac{\boldsymbol{p}}{\mathcal{E}} \times \boldsymbol{s}\right) \times \boldsymbol{E} + \boldsymbol{s} \times \boldsymbol{B}\right). \tag{22c}$$

where

$$\mathcal{E} = \sqrt{|\boldsymbol{p}|^2 - 2(eg/2)\boldsymbol{s}\cdot\boldsymbol{B}} \;\neq |\boldsymbol{p}|.$$

N.B. For weak, pure magnetic field, $\boldsymbol{s} = \frac{1}{2}\widehat{\boldsymbol{p}}$:

$$\mathcal{E} \approx |\boldsymbol{p}| - \frac{eg}{4} S\cdot F = |\boldsymbol{p}| - e\,\frac{\widehat{\boldsymbol{p}}\cdot\boldsymbol{B}}{2|\boldsymbol{p}|}, \tag{23}$$

modif dispersion relation, cf.

Son-Yamamoto, Manuel-Torres-Rincon, Chen-Son-Stephanov-Yee-Yin.

N.B. : No anomalous term on rhs of (22a).

In pure magnetic field momentum and spin satisfy eqns of identical form,

$$\frac{d\boldsymbol{p}}{dt} = \frac{e}{\mathcal{E}}\boldsymbol{p} \times \boldsymbol{B}, \quad \frac{d\boldsymbol{s}}{dt} = \frac{e}{\mathcal{E}}\boldsymbol{s} \times \boldsymbol{B}. \tag{24}$$

Thus

$$\begin{cases} |\boldsymbol{p}| = \text{const} \neq 0, & \boldsymbol{p} \cdot \boldsymbol{B} = \text{const}, \\ |\boldsymbol{s}| = \text{const} \neq 0, & \boldsymbol{s} \cdot \boldsymbol{B} = \text{const}, \end{cases} \Longrightarrow$$

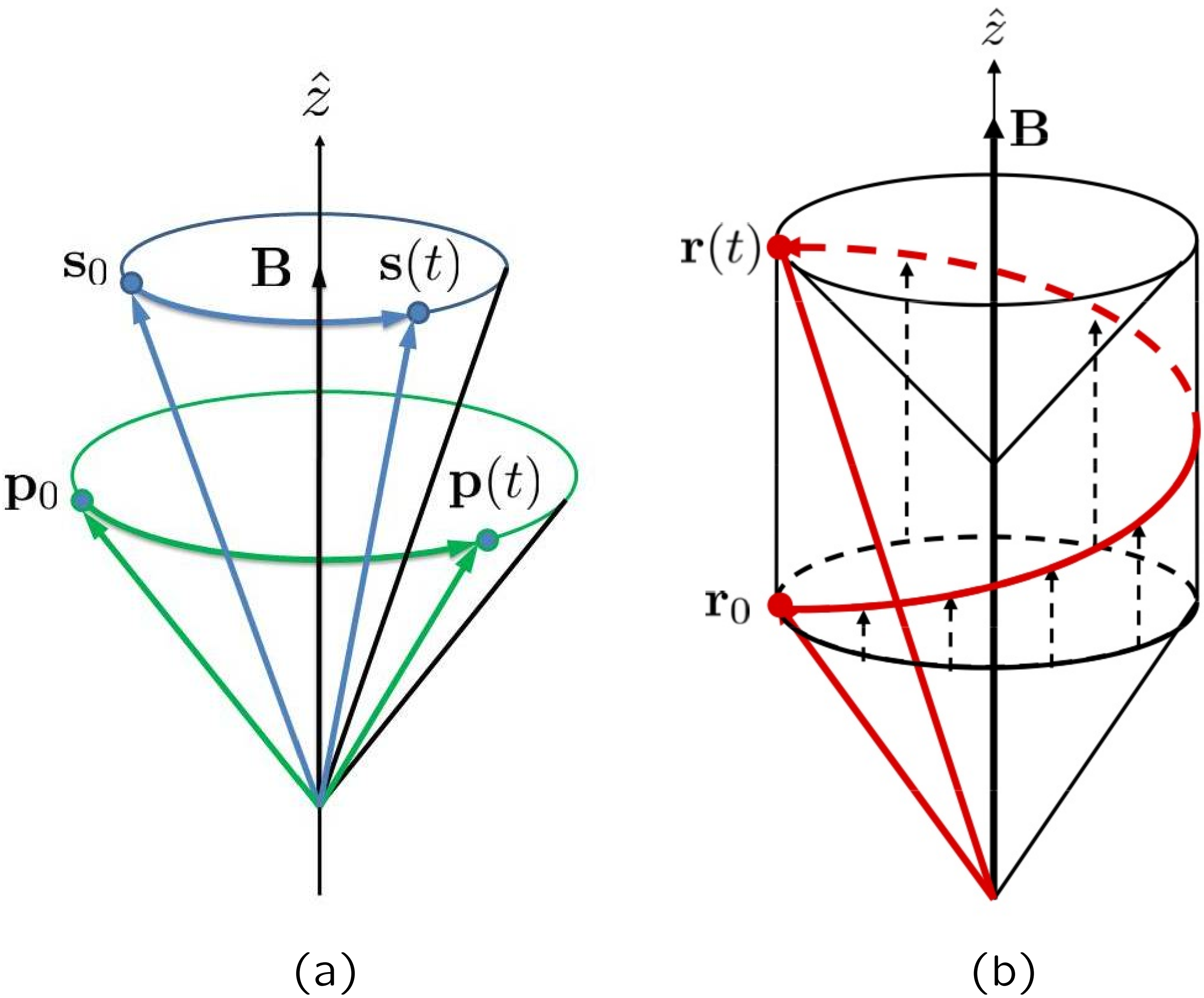
$$\begin{cases} p_z = \text{const}, & s_z = \text{const}, \\ \mathcal{E} = \sqrt{|\boldsymbol{p}|^2 - e\boldsymbol{s} \cdot \boldsymbol{B}} = \text{const}. \end{cases} \tag{25}$$

$$\boldsymbol{p}(t) = (p_0 e^{-i(eB/\mathcal{E})t}, p_z), \tag{26a}$$

$$\boldsymbol{s}(t) = (s_0 e^{-i(eB/\mathcal{E})t}, s_z), \tag{26b}$$

$$\mathbf{r}(t) = \left(\frac{ip_0}{eB} e^{-i(eB/\mathcal{E})t}, \frac{p_z}{\mathcal{E}} t\right) + \mathbf{r}_0, \tag{26c}$$

where $p_0 = p_x + ip_y$, $s_0 = s_x + is_y$.



(a) Motion in pure constant magnetic field $\boldsymbol{B}$*. Both momentum* $\boldsymbol{p}(t)$ *and spin* $\boldsymbol{s}(t)$ *precess with identical angular velocity* $\omega = -eB/\mathcal{E}$ *around* $\boldsymbol{B}$*. (b) Position* $\mathbf{r}(t)$ *spirals on cylinder around* $\boldsymbol{B}$*, combining precession with vertical drift of the supporting cone itself.*

N.B. Some authors advocate using "Newton-Wigner-Pryce" coordinate

$$\tilde{\boldsymbol{x}} = \mathbf{r} + \frac{\widehat{\boldsymbol{p}} \times \boldsymbol{s}}{|\boldsymbol{p}|} \tag{27}$$

they identify with "center of particle".

N.B. components of Pryce coord *not* commute.

In free case $\tilde{\boldsymbol{x}}$ is conserved & can be used to label motion [1].

Using (27), spin becomes "enslaved" to momentum (as in c-model), i.e., ang. mom is

$$\boldsymbol{\ell} = \tilde{\boldsymbol{x}} \times \boldsymbol{p} + \boxed{s\,\widehat{\boldsymbol{p}}}\,. \tag{28}$$

Velocity relation (22a) acquires anomalous terms even for $g = 2$,

$$\frac{d\tilde{\boldsymbol{x}}}{dt} = \frac{\boldsymbol{p}}{\mathcal{E}} - e\frac{\boldsymbol{s} \times \boldsymbol{E}}{|\boldsymbol{p}|^2} \tag{29}$$

$$-e(\widehat{\boldsymbol{p}} \cdot \boldsymbol{E})\Big(\frac{2\mathcal{E}^2 - |\boldsymbol{p}|^2}{\mathcal{E}^2|\boldsymbol{p}|}\Big)\frac{\widehat{\boldsymbol{p}} \times \boldsymbol{s}}{|\boldsymbol{p}|} + \frac{e}{\mathcal{E}}\Big(\frac{\widehat{\boldsymbol{p}} \times \boldsymbol{s}}{|\boldsymbol{p}|}\Big) \times \boldsymbol{B}.$$

Eqns (22) (reminiscent of those of massive relativistic particle) $\Rightarrow$ *spin-field term is const. of motion* *. Yields also time-variation of $\mathcal{E}$,

$$\frac{d}{dt}(S.F) = 0, \qquad \frac{d}{dt}(\mathcal{E}^2) = 2e\,\boldsymbol{p}\cdot\boldsymbol{E}. \tag{30}$$

Therefore

$$\mathcal{E} - e\mathbf{r}\cdot\boldsymbol{E}\,, \tag{31}$$

conserved, interpreted as (field-spin-dependent) energy. NB: $\mathcal{E} \neq |\boldsymbol{p}|$ & alone *not* conserved.

Using eqns of motion (22) find, for $\boxed{g = 2}$,

$$\boxed{\frac{d\chi^{\circ}}{dt} = e^2\Big(\frac{S.F}{\mathcal{E}^2|\boldsymbol{p}|}\Big)\,\boldsymbol{s}\cdot\Big(\widehat{\boldsymbol{p}}\times(\widehat{\boldsymbol{p}}\times\boldsymbol{E})\Big),} \tag{32}$$

$\Rightarrow$ helicity χ° may *not* be conserved.

Below present example, where helicity is indeed not const of motion.

*Statement holds for any value of g provided the fields are constant.

Motion in pure electric field for $g = 2$

For $\boldsymbol{B} = 0$ $g = 2$ eqns. (22) reduce to

$$\mathcal{E}\,\frac{d\mathbf{r}}{dt} = \boldsymbol{p}, \quad \frac{d\boldsymbol{p}}{dt} = e\boldsymbol{E}, \quad \frac{d\boldsymbol{s}}{dt} = \frac{e}{\mathcal{E}^2}\big(\boldsymbol{p}\times\boldsymbol{s}\big)\times\boldsymbol{E}, \quad (33)$$

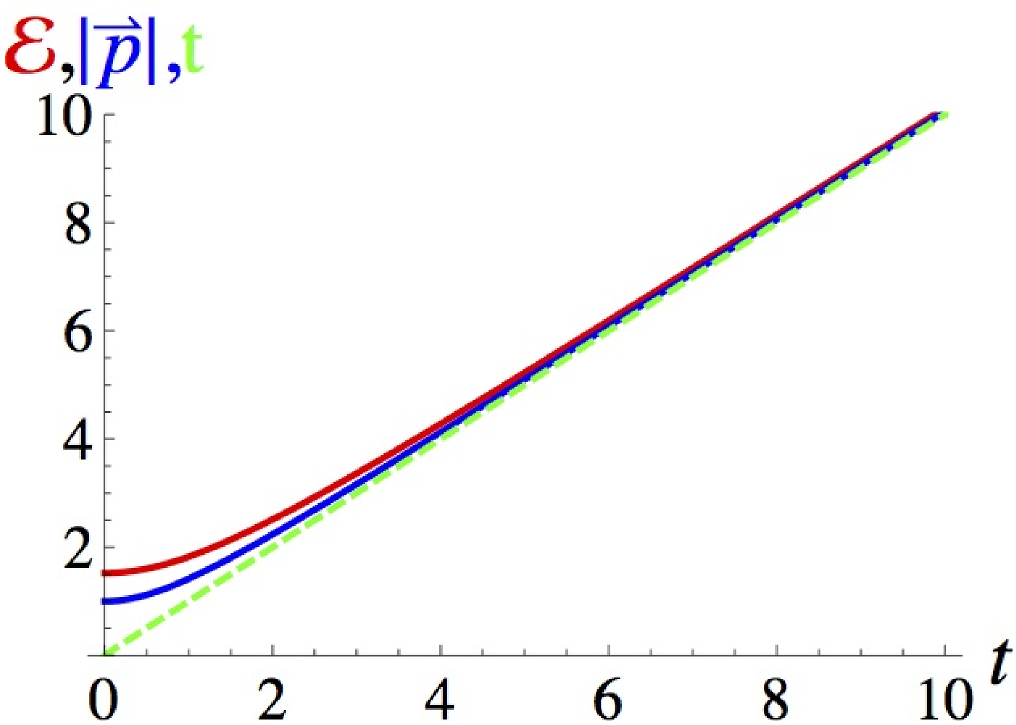
with

$$\mathcal{E} = \sqrt{|\boldsymbol{p}|^2 + 2e\,\boldsymbol{s}\cdot\big(\frac{\boldsymbol{p}}{\mathcal{E}}\times\boldsymbol{E}\big)}\,. \qquad (34)$$

Momentum eqn integrated as $\boldsymbol{p}(t) = \boldsymbol{p}_0 + e\boldsymbol{E}t$
$\Rightarrow$ from (30) infer

$$\mathcal{E}^2(t) = \mathcal{E}_0^2 + 2e(\boldsymbol{p}_0\cdot\boldsymbol{E})t + e^2\boldsymbol{E}^2t^2.$$

($\Rightarrow$ for large t, $\mathcal{E} \sim e|E|t$, consistently with conservation of $S.F$ and with $|\boldsymbol{p}| \sim e|E|t$).



$\boldsymbol{E}$-component of spin is conserved,

$$d(\boldsymbol{s}\cdot\boldsymbol{E})/dt = 0.$$

Combining results,

$$\frac{d\boldsymbol{s}}{dt} = -\frac{e}{\mathcal{E}^2(t)}\Big((\boldsymbol{s}_0\cdot\boldsymbol{E})(\boldsymbol{p}_0+e\boldsymbol{E}t)-(\boldsymbol{p}_0\cdot\boldsymbol{E}+e\boldsymbol{E}^2t)\boldsymbol{s}\Big).$$

Explicitly time-dependent 1st order diff. eqn

$$\boldsymbol{s}(t) \quad = \frac{\boldsymbol{s}_0\cdot\boldsymbol{E}}{\boldsymbol{E}^2\mathcal{E}_0^2-(\boldsymbol{p}_0\cdot\boldsymbol{E})^2}\times$$

$$\Big(\mathcal{E}_0^2\boldsymbol{E}+e(\boldsymbol{p}_0\cdot\boldsymbol{E})\boldsymbol{E}t-(\boldsymbol{p}(t)\cdot\boldsymbol{E})\boldsymbol{p}_0\Big)+\mathcal{E}(t)\,\boldsymbol{a}, \qquad (35)$$

where const vector $\boldsymbol{a}$ determined by init conds

$$\boldsymbol{p}(0)=\boldsymbol{p}_0, \quad \boldsymbol{s}(0)=\boldsymbol{s}_0, \quad \mathbf{r}(0)=\mathbf{r}_0.$$

Eqn. (35) $\Rightarrow$ $\boldsymbol{a}$ perpendicular to electric field, $\boldsymbol{a}\cdot\boldsymbol{E}=0$.

Spatial motion dealt with similarly,

$$\mathbf{r}(t) = \mathbf{r}_0 \;+\; \frac{\Big(\boldsymbol{E}\times(\boldsymbol{p}\times\boldsymbol{E})\Big)}{e|\boldsymbol{E}|^3}\log\left|\left\{\frac{\boldsymbol{p}\cdot\boldsymbol{E}+\mathcal{E}|\boldsymbol{E}|}{\boldsymbol{p}_0\cdot\boldsymbol{E}+\mathcal{E}_0|\boldsymbol{E}|}\right\}\right|$$

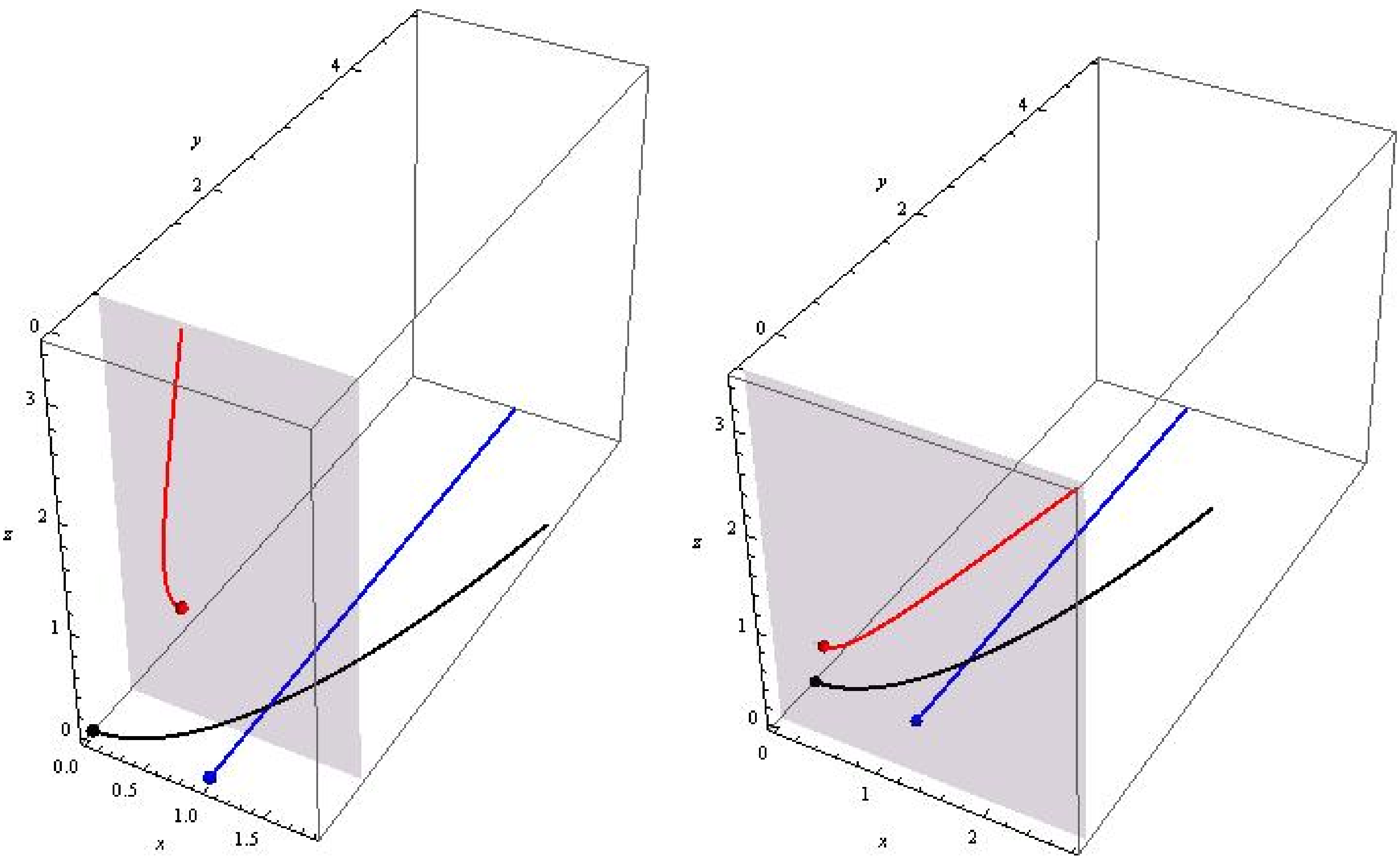
$$+\;\frac{\boldsymbol{E}}{e\boldsymbol{E}^2}(\mathcal{E}-\mathcal{E}_0)\,. \qquad (36)$$

Eqns (33) also studied numerically.

$\mathcal{E}(t)$ satisfies 3rd-order algebraic eqn

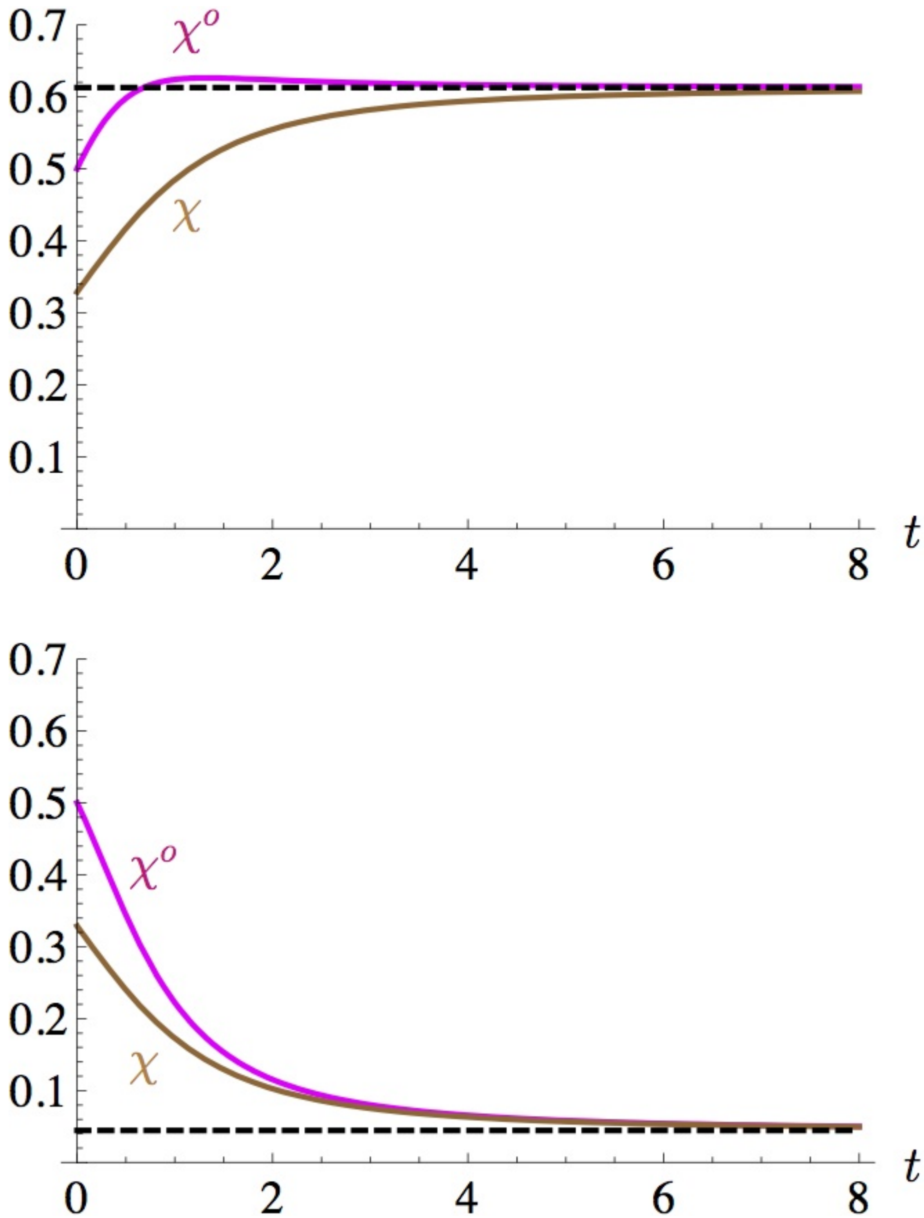
$$\mathcal{E}^2 = |\boldsymbol{p}|^2 + 2e\,\boldsymbol{s}\cdot\left(\frac{\boldsymbol{p}}{\mathcal{E}}\times\boldsymbol{E}\right)$$

$\rightsquigarrow$ solved for each $\boldsymbol{p}(t)$ and thus for each t using Cardano's formula; remaining eqns integrated numerically. $\boldsymbol{E} = E\widehat{\boldsymbol{y}}$ with init conds $\mathbf{r}(0) = 0$, $\boldsymbol{p}(0) = \boldsymbol{p}_0 = p_0\widehat{\boldsymbol{x}}$, $\boldsymbol{s}(0) = \boldsymbol{s}_0$.



3-dimensional motion ($g = 2$). **Spin** *moves in the vertical plane $s_y = \pm\frac{1}{2}$ whereas* **momentum** *and* **position** *move in horizontal planes. Spin not enslaved.*

Helicity χ° (in purple) is initially *not* conserved,



For spin-extended massless particle with $g = 2$ *put into an electric field, neither of helicities* $\chi^{\circ} = \boldsymbol{s} \cdot \widehat{\boldsymbol{p}}$ *nor* $\chi = \boldsymbol{s} \cdot \boldsymbol{p}/\mathcal{E}$ *conserved. Both expressions converge asymptotically to common value.*

Helicity, revisited

Gyromagnetic factor g : any real number (unless contrary is said).

Constraint $S_{\mu\nu}S^{\mu\nu} = 2s^2$ in (12) $\Rightarrow$

$$\boldsymbol{s}^2 - \frac{1}{\mathcal{E}^2}\Big(|\boldsymbol{p}|^2|\boldsymbol{s}|^2 - (\boldsymbol{p}\cdot\boldsymbol{s})^2\Big) = s^2 . \qquad (37)$$

For $g = 0$ general dispersion relation (14) reduces to $\mathcal{E} = |\boldsymbol{p}|$, (37) $\Rightarrow$ helicity constraint $\widehat{\boldsymbol{p}}\cdot\boldsymbol{s} = \pm|s|$ in (19)

However for $g \neq 0$ this is *not* so $\rightsquigarrow$ revisit & definition of helicity.

Start again with free case. Remember: space of motions is coadjoint orbit of Poincaré group, labeled by Casimir invariants m (mass) and s (spin).

Decomposing Lorentz & translational moments, $M = (M^{\mu\nu}) \in \mathfrak{o}(3,1)$ resp. $P = (P^{\mu}) \in \mathbf{R}^{3,1}$, into orbital & spin constituents,

$$M^{\mu\nu} = R^{\mu}P^{\nu} - R^{\nu}P^{\mu} + S^{\mu\nu}. \qquad (38)$$

In Lorentz frame Casimirs

$$m^2 = -\boldsymbol{p}^2 + \mathcal{E}^2 \qquad s = \boldsymbol{\ell} \cdot \widehat{\boldsymbol{p}}\,, \tag{39}$$

where $\boldsymbol{\ell}$, angular momentum, is space component of M, $M_{ij} = \epsilon_{ijk}\ell_k$; real number s is scalar spin.

Pauli-Lubanski vector :

$$W_\sigma = \tfrac{1}{2}\,\epsilon_{\mu\nu\rho\sigma} M^{\mu\nu} P^\rho. \tag{40}$$

For non-tachyonic coadjoint orbits, $m \geq 0$ and

$$W_\mu W^\mu = -s^2 m^2. \tag{41}$$

Orbits we are interested in are massless, $m = 0$ and have scalar spin $|s| > 0$ (commonly $|s| = \frac{1}{2}$). Null & orthogonal (thus parallel) vectors W and P are proportional,

$$\boxed{W = \chi\, P} \tag{42}$$

defines helicity as proportionality factor between W^μ and P^μ. In Lorentz frame,

$$W = \chi \begin{pmatrix} \boldsymbol{p} \\ \mathcal{E} \end{pmatrix}, \quad \chi = \boldsymbol{s} \cdot \frac{\boldsymbol{p}}{\mathcal{E}}\,. \tag{43}$$

In free case $\mathcal{E} = |\boldsymbol{p}|$ & recover old expression in (19), $\chi = \chi^{\mathrm{o}} = \pm|s|$.

Conservation also checked using (free) eqns of motion, is consistent with group theory, for which it is Casimir invariant.

In coupled case, (12), similar calculation $\rightsquigarrow$

$$W = \boldsymbol{s} \cdot \frac{\boldsymbol{p}}{\mathcal{E}} \begin{pmatrix} \boldsymbol{p} \\ \mathcal{E} \end{pmatrix} - \boxed{\frac{(eg/2)S.F}{\mathcal{E}} \begin{pmatrix} \boldsymbol{s} \\ 0 \end{pmatrix}}, \qquad (44)$$

$\Rightarrow$ Pauli-Lubanski & momentum no longer parallel unless effective mass vanishes.

New helicity in (43), appears as coefficient of first term. Is it conserved ? Rewriting (37) as

$$\chi^2 = s^2 + (eg/2)\frac{S.F}{\mathcal{E}^2}\,|\boldsymbol{s}|^2$$

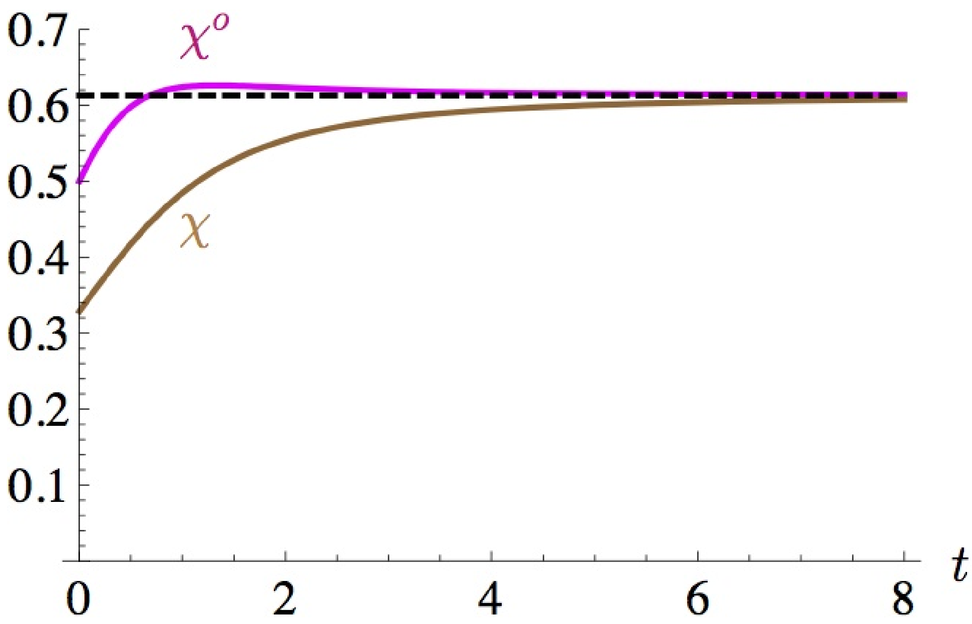
$\Rightarrow$ (43) conserved for minimal coupling, $g = 0$ but not for $g \neq 0$, because extra term not const of motion.

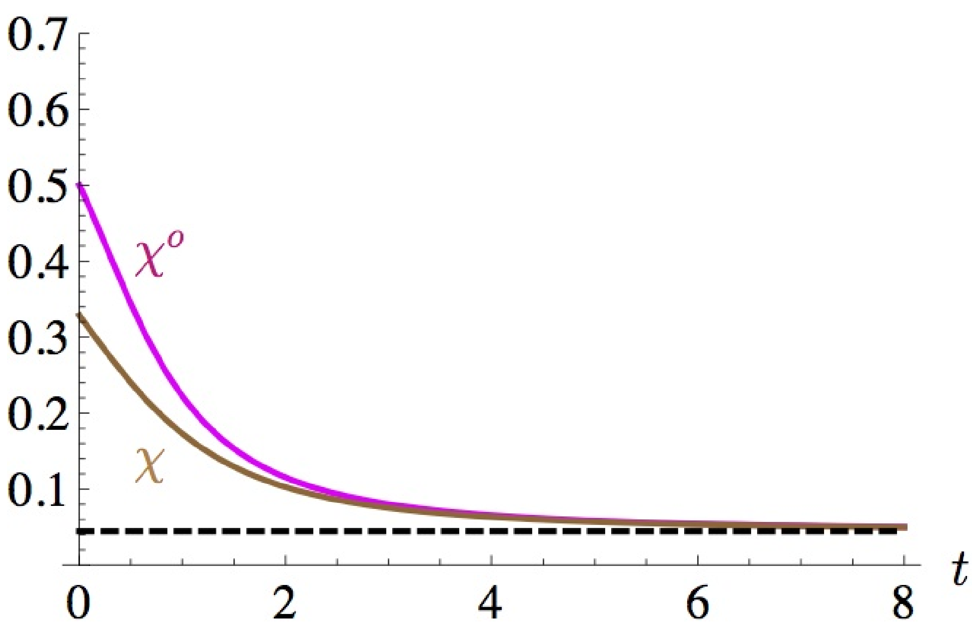
For $g = 2 \Rightarrow$ (32) should be replaced by

$$\frac{d\chi}{dt} = -\left(\frac{e^2\, S.F}{\mathcal{E}^3}\right)\, \boldsymbol{s} \cdot \boldsymbol{E} \;\neq 0 \qquad (45)$$

$$W_\mu W^\mu = s^2\,\frac{eg}{2}\,S.F \;= s^2 m^2 \qquad (46)$$

cf. massive case, eqn. (41).





For spin-extended massless particle ($g = 2$) in electric field, ***neither*** *of helicities $\chi^{\circ} = \boldsymbol{s} \cdot \widehat{\boldsymbol{p}}$ nor $\chi = \boldsymbol{s} \cdot \boldsymbol{p}/\mathcal{E}$ conserved. Both expressions converge to common value asymptotically.* $\Rightarrow$ both helicities χ° & χ tend to same const value when $t \to \infty$. Helicity ***not*** conserved only during short initial period.

Transport properties (c-model)

Invariant volume element

$$dV_c = D_c\, d^3\boldsymbol{p} d^3\boldsymbol{x}, \quad D_c = 1 + e\boldsymbol{\Theta}\cdot\boldsymbol{B}. \quad (47)$$

$D_c = \sqrt{\det(\omega_c)}$ determinant of symplectic matrix ω_c. System regular if $D_c \neq 0$.

Liouville's thm takes anomalous form

$$\frac{\partial D_c}{\partial t} + \frac{\partial(D_c\dot{\boldsymbol{x}})}{\partial \boldsymbol{x}} + \frac{\partial(D_c\,\dot{\boldsymbol{p}})}{\partial \boldsymbol{p}} = (\boldsymbol{E}\cdot\boldsymbol{B})\,\boldsymbol{\nabla}_{\boldsymbol{p}}\cdot\left(\frac{\widehat{\boldsymbol{p}}}{2|\boldsymbol{p}|^2}\right)$$

$$= 2\pi e^2(\boldsymbol{E}\cdot\boldsymbol{B})\,\delta^3(\boldsymbol{p}). \quad (48)$$

$f(\boldsymbol{x},\boldsymbol{p},t)$ distribution assumed satisfy collisionless Boltzmann eqn $\partial_t f + \partial_{\boldsymbol{x}} f\,\dot{\boldsymbol{x}} + \partial_{\boldsymbol{p}} f\,\dot{\boldsymbol{p}} = 0$.

Current density

$$\boldsymbol{j} \;=\; \int D_c \frac{d^3\boldsymbol{p}}{(2\pi)^3} f\,\dot{\boldsymbol{x}} \;=\; \underbrace{\int \frac{d^3\boldsymbol{p}}{(2\pi)^3}\, f\widehat{\boldsymbol{p}}}_{normal\ \ current} \tag{49}$$

$$+\;\underbrace{e\boldsymbol{B}\int\frac{d^3\boldsymbol{p}}{(2\pi)^3}\frac{f}{2|\boldsymbol{p}|^2}}_{chiral\ \ magnetic} + \underbrace{e\boldsymbol{E}\times\int\frac{d^3\boldsymbol{p}}{(2\pi)^3}\frac{f\widehat{\boldsymbol{p}}}{2|\boldsymbol{p}|^2}}_{anom\ \ Hall\ \ curr}. \tag{50}$$

Particle density

$$\rho(\boldsymbol{x},t) = \int f D_c \frac{d^3\boldsymbol{p}}{(2\pi)^3} \qquad \rightsquigarrow$$

anomalous continuity eqn (chiral anomaly),

$$\partial_t \rho + \boldsymbol{\nabla}\cdot\boldsymbol{j} = \frac{e^2}{4\pi^2}(\boldsymbol{E}\cdot\boldsymbol{B})\, f(0)\,. \tag{51}$$

Anomalies in S-model [2]

System regular when

$$D_S \equiv \sqrt{\det(\omega_S)} = \frac{e\,\widehat{\boldsymbol{p}}\cdot\boldsymbol{B}}{|\boldsymbol{p}|^2\,\widehat{p}_3} = \frac{eB}{|\boldsymbol{p}|^2} \neq 0\,. \qquad (52)$$

Liouville's thm

$$\frac{\partial D_S}{\partial t} + \frac{\partial(D_S\,\dot{\mathbf{r}})}{\partial \mathbf{r}} + \frac{\partial(D_S\,\dot{\boldsymbol{p}})}{\partial \boldsymbol{p}} + \frac{\partial(D_S\,\dot{s}_a)}{\partial s_a} =$$

$$e^2(\boldsymbol{E}\cdot\boldsymbol{B})\,\boldsymbol{\nabla}_{\boldsymbol{p}}\cdot\Big(\frac{\widehat{\boldsymbol{p}}}{|\boldsymbol{p}|^2\widehat{p}_3}\Big) = e^2\,\frac{(\boldsymbol{E}\cdot\boldsymbol{B})}{\widehat{p}_3}\,4\pi\delta^3(\boldsymbol{p}).$$

Particle current,

$$\boldsymbol{j}(\mathbf{r},t) = \int f\,\dot{\mathbf{r}}\,D_S d^3\boldsymbol{p}\,ds_1 ds_2 = \qquad (53)$$

$$\underbrace{e\boldsymbol{B}\int\frac{f}{|\boldsymbol{p}|^2\widehat{p}_3}d^3\boldsymbol{p}\,ds_1 ds_2}_{CME} + \underbrace{e\boldsymbol{E}\times\int\frac{f\widehat{\boldsymbol{p}}}{|\boldsymbol{p}|^2\widehat{p}_3}d^3\boldsymbol{p}\,ds_1 ds_2}_{anom\;\;Hall}$$

Particle density $\rho(\mathbf{r},t) = \int f D_S d^3\boldsymbol{p}\,ds_1 ds_2,$

$$\partial_t\rho(\mathbf{r},t) = \int\Big(\frac{\partial D_S}{\partial t}\Big) f d^3\boldsymbol{p} ds_1 ds_2 + \int D_S\frac{\partial f}{\partial t} d^3\boldsymbol{p} ds_1 ds_2.$$

Dropping boundary terms,

$$\partial_t\rho(\mathbf{r},t) + \boldsymbol{\nabla}_{\mathbf{r}}\cdot\boldsymbol{j}(\mathbf{r},t) = \qquad (54)$$

$$e^2\boldsymbol{E}\cdot\boldsymbol{B}\int f\boldsymbol{\nabla}_{\boldsymbol{p}}\cdot\Big(\frac{\widehat{\boldsymbol{p}}}{|\boldsymbol{p}|p_3}\Big)d^3\boldsymbol{p} ds_1 ds_2 = e^2\boldsymbol{E}\cdot\boldsymbol{B}\,\frac{4\pi\,f_0}{\widehat{p}_3}.$$