On Entanglement Hamiltonians in one-dimensional quantum systems

Erik Tonni

SISSA

SISSA

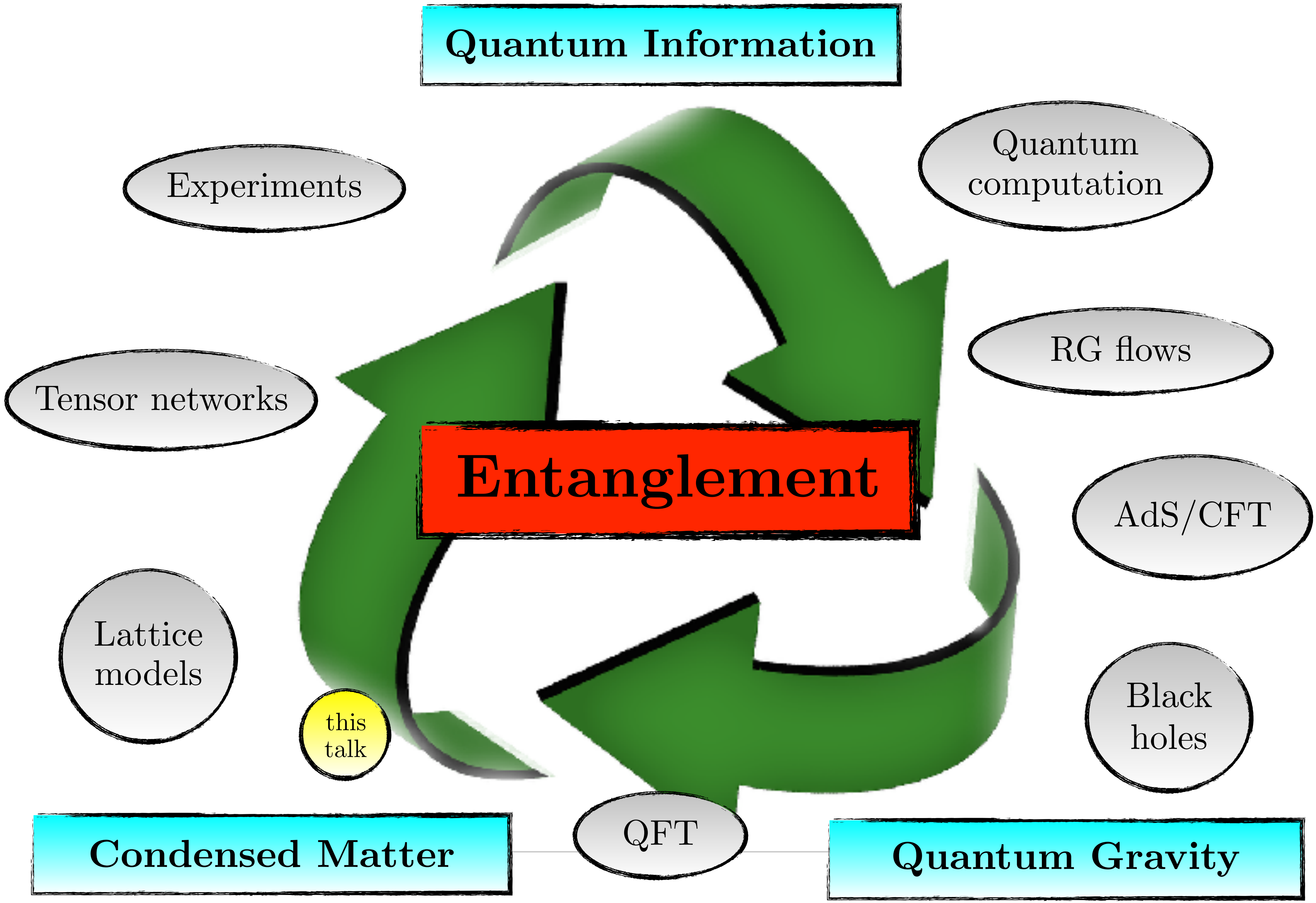
INFN Istituto Nazionale di Fisica Nucleare

Giuseppe Di Giulio, E.T.	[1911.07188] **JSTAT**
Jacopo Surace, Luca Tagliacozzo, E.T.	[1909.07381] **PRB(R)** to appear
Giuseppe Di Giulio, Raul Arias, E.T.	[1905.01144] **JSTAT**
Viktor Eisler, E.T., Ingo Peschel	[1902.04474] **JSTAT**
John Cardy, E.T.	[1608.01283] **JSTAT**

Exploration of Duality, Geometry and Entanglement

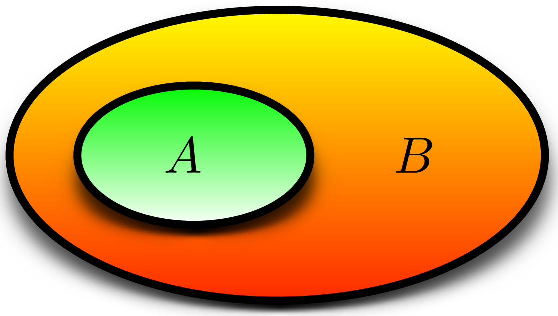
LE STUDIUM Consortium, June 4th 2020, virtual meeting

Entanglement: a crossroad of interests



Entanglement Entropy (EE), EH & Contour for EE

- Quantum system in its ground state: $\rho = |\Psi\rangle\langle\Psi|$
  Factorised Hilbert space $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$



e.g.: spatial bipartition

- A's **reduced density matrix** $\rho_A = \text{Tr}_B \rho$

  (normalisation: $\text{Tr}\rho_A = 1$)

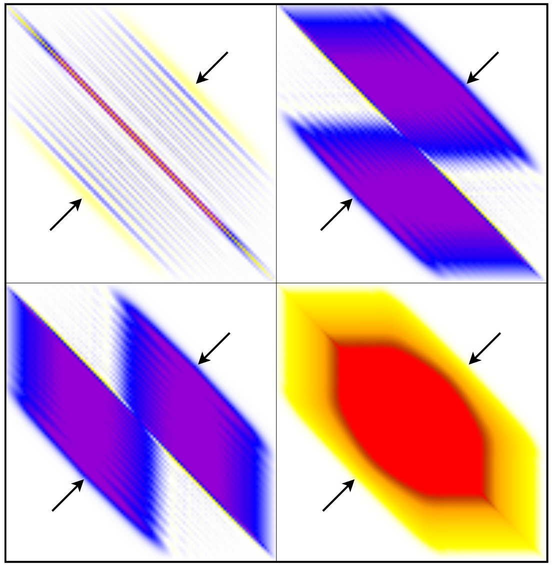
- Entanglement Entropy (EE) $S_A \equiv -\text{Tr}_A(\rho_A \log \rho_A) = -\partial_n \text{Tr}_A(\rho_A^n)\big|_{n=1}$

- $\rho_A \propto e^{-\widehat{K}_A}$ Entanglement Hamiltonian (EH) $\widehat{K}_A$

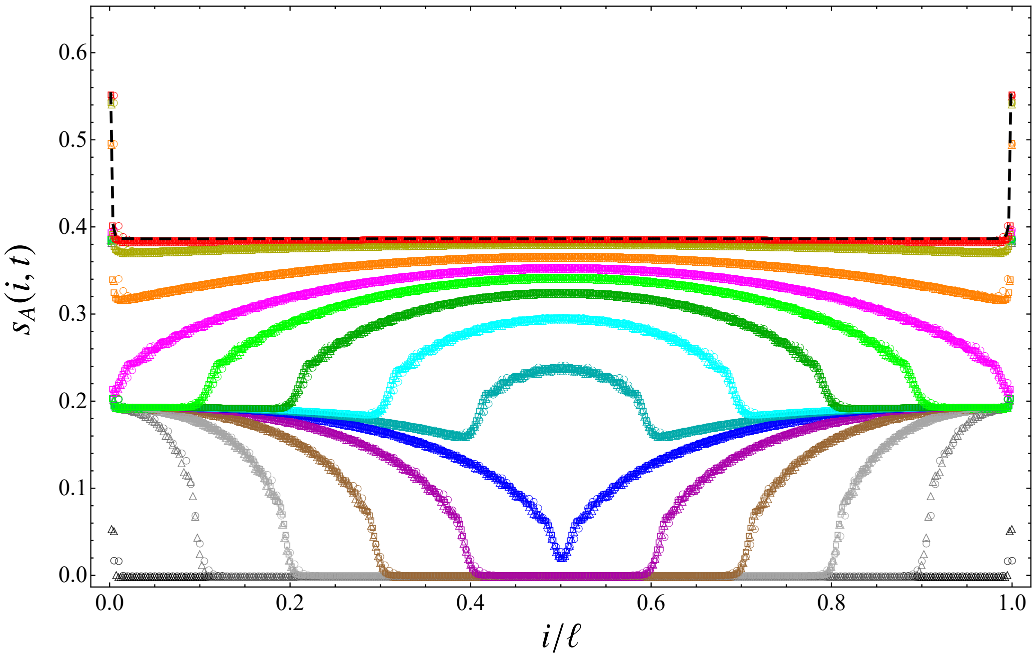
- $S_A = \sum_{i \in A} s_A(i)$ Contour function for the EE $s_A(i)$

  $s_A(i) \geqslant 0$ (and other requirements)

Outline



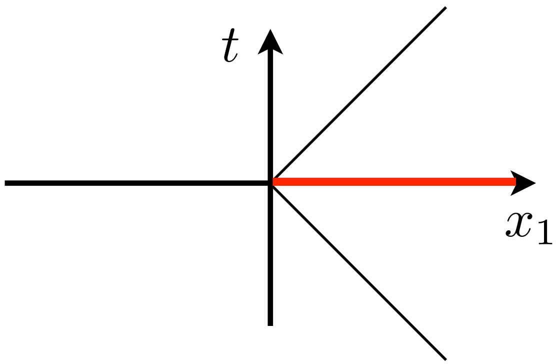
- Entanglement Hamiltonians (EH) in CFT
  [Cardy, E.T., (2016)]
- EH and contours for the EE in
  - harmonic chains (HC)
  - free fermions chains (FFC)
- Continuum limit of the EH of an interval in a FFC and in the HC
  [Eisler, E.T., Peschel, (2019)] [Di Giulio, E.T., (2019)]
- Global quantum quenches: temporal evolutions of
  - EH matrices
  - Contours for the EE
- Insights from CFT and from the quasi-particle picture
  [Di Giulio, Arias, E.T., (2019)]
  [Surace, Tagliacozzo, E.T., (2019)]



EH in field theories: Bisognano-Wichmann & CFT

- QFT in its ground state, A is the (right) half-space $x_1 > 0$ in $\mathbb{R}^{1,d-1}$

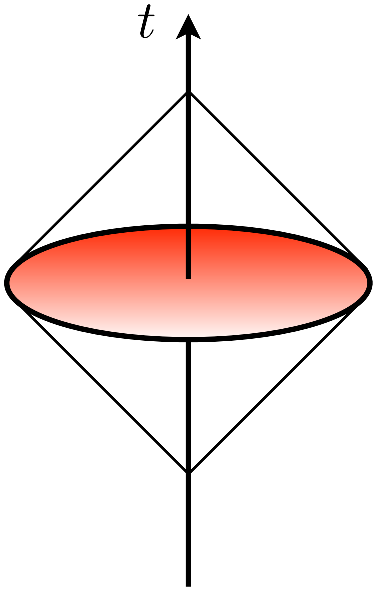
$$\widehat{K}_A = \int_A x_1 \, T_{00} \, d^{d-1}x$$



$\widehat{K}_A$ is the generator of the Lorentz boosts in the (right) Rindler wedge [Bisognano, Wichmann, (1975)]

- CFT_d and A is a ball of radius R

$$\widehat{K}_A = 2\pi \int_A \frac{R^2 - r^2}{2R} \, T_{00} \, d^{d-1}x$$



The Rindler wedge can be conformally mapped into the causal diamond of A

[Hislop, Longo, (1982)] [Casini, Huerta, Myers, (2011)]

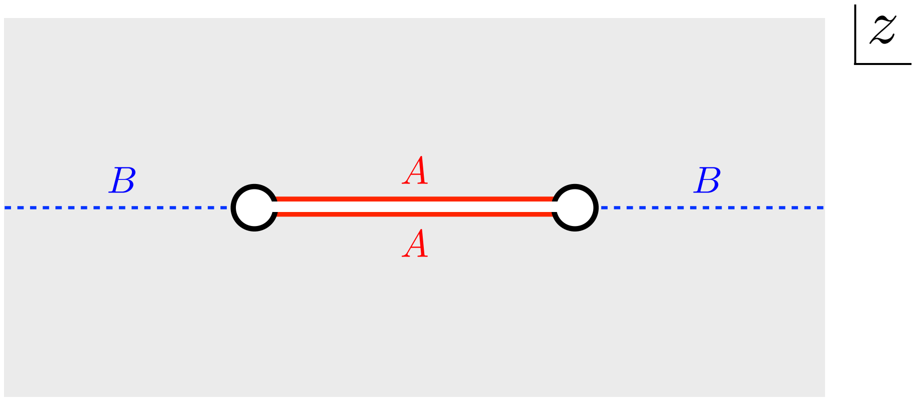
- CFT_2 and A is an interval (ground state)

$$\widehat{K}_A = 2\pi \int_0^\ell \frac{x(\ell - x)}{\ell} \, T_{00} \, dx$$

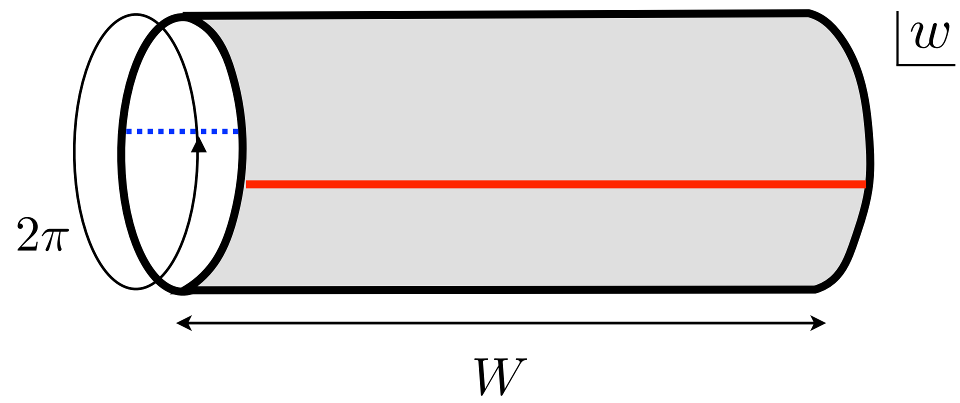
Mapping to the annulus: EE & Entanglement Spectrum

[Cardy, E.T., (2016)]

Reduced density matrix



Partition function on the annulus



- Rényi entropies

$$\mathrm{Tr}\rho_A^n = \frac{\mathcal{Z}_{n\text{-annulus}}}{(\mathcal{Z}_{\text{annulus}})^n}$$

Modular parameter of the annulus $q \equiv e^{-2\pi^2/W}$

- For $W \gg 1$

$$S_A^{(n)} \sim \frac{c}{12}\left(1+\frac{1}{n}\right)W + g_a + g_b$$

boundary entropies $g_{a,b} = -\log\langle a,b|0\rangle$

- *Entanglement Spectrum*: Eigenvalues of $\widehat{K}_A$

$$-\log\lambda_j = \frac{c}{12}W_A + \log\big(a|0\rangle\langle 0|b\rangle\big) + \left(\Delta_j - \frac{c}{24}\right)\frac{2\pi^2}{W_A} + O(\epsilon^r)$$

$$\frac{g_r}{g_1} = \frac{\Delta_r}{\Delta_1}$$

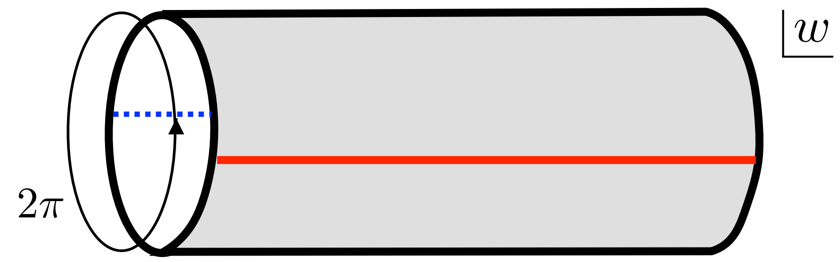
Δ_j conformal dimensions (primaries and descendants) of the BCFT

Mapping to the annulus: Entanglement Hamiltonian

- The entanglement hamiltonian K_A is then the conformal image of the generator of translations around the annulus in the direction of $v = \mathrm{Im}\, w$

In the $w = u + iv$ and z domains we have respectively

$$K_A = -\int_{v=\mathrm{const}} T_{vv}\, du$$



$$K_A = \int_C \frac{T(z)}{f'(z)}\, dz + \int_{\overline{C}} \frac{\overline{T}(\bar{z})}{\overline{f'(z)}}\, d\bar{z}$$

The schwartzian term does not contribute in the ratio in $\mathrm{Tr}\rho_A^n$

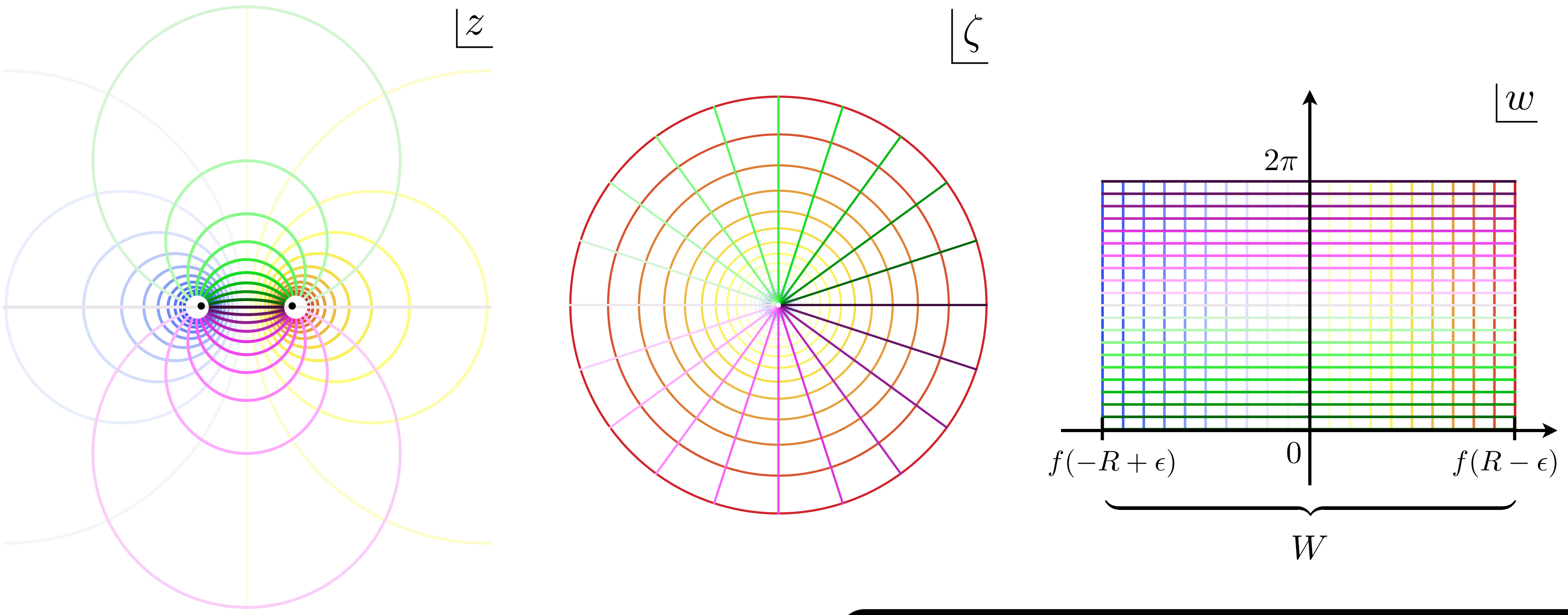
- For almost all the known time-independent cases

$$K_A = \int_A \frac{T_{00}(x)}{f'(x)} dx$$

Entanglement hamiltonians: time-independent cases I

[Läuchli, (2013)] [Wong, Klich, Pando-Zayas, Vaman, (2013)] [Ohmori, Tachikawa, (2015)] [Cardy, E.T., (2016)]

- Ground state and single interval $A = (-R, +R)$ on the infinite line



$$w = f(z) = \log \zeta = \log\left(\frac{z+R}{R-z}\right)$$

$$K_A = \int_A \frac{R^2 - x^2}{2R}\, T_{00}(x)\, dx$$

$$S_A^{(n)} = \frac{c}{6}\left(1 + \frac{1}{n}\right)\log(\ell/\epsilon) + g_a + g_b + \text{corrections}$$

Entanglement hamiltonians: time-independent cases II

[Wong, Klich, Pando-Zayas, Vaman, (2013)] [Cardy, E.T., (2016)]

- Ground state and finite interval $A = (-R, R)$ in a finite spatial circle

$$f(z) = \log\left(\frac{e^{2\pi \mathrm{i} z/L} - e^{-2\pi \mathrm{i} R/L}}{e^{2\pi \mathrm{i} R/L} - e^{2\pi \mathrm{i} z/L}}\right)$$

$$K_A = \frac{L}{\pi}\int_A \frac{\sin[\pi(R-x)/L]\,\sin[\pi(x+R)/L]}{\sin(2\pi R/L)}\,T_{00}(x)\,dx$$

$$S_A^{(n)} = \frac{c}{6}\left(1+\frac{1}{n}\right)\log\left(\frac{L}{\pi\epsilon}\sin(\pi\ell/L)\right) + g_a + g_b + \text{corrections}$$

- Finite interval $(-R, R)$ in an infinite system at finite temperature

$$f(z) = \log\left(\frac{e^{2\pi z/\beta} - e^{-2\pi R/\beta}}{e^{2\pi R/\beta} - e^{2\pi z/\beta}}\right)$$

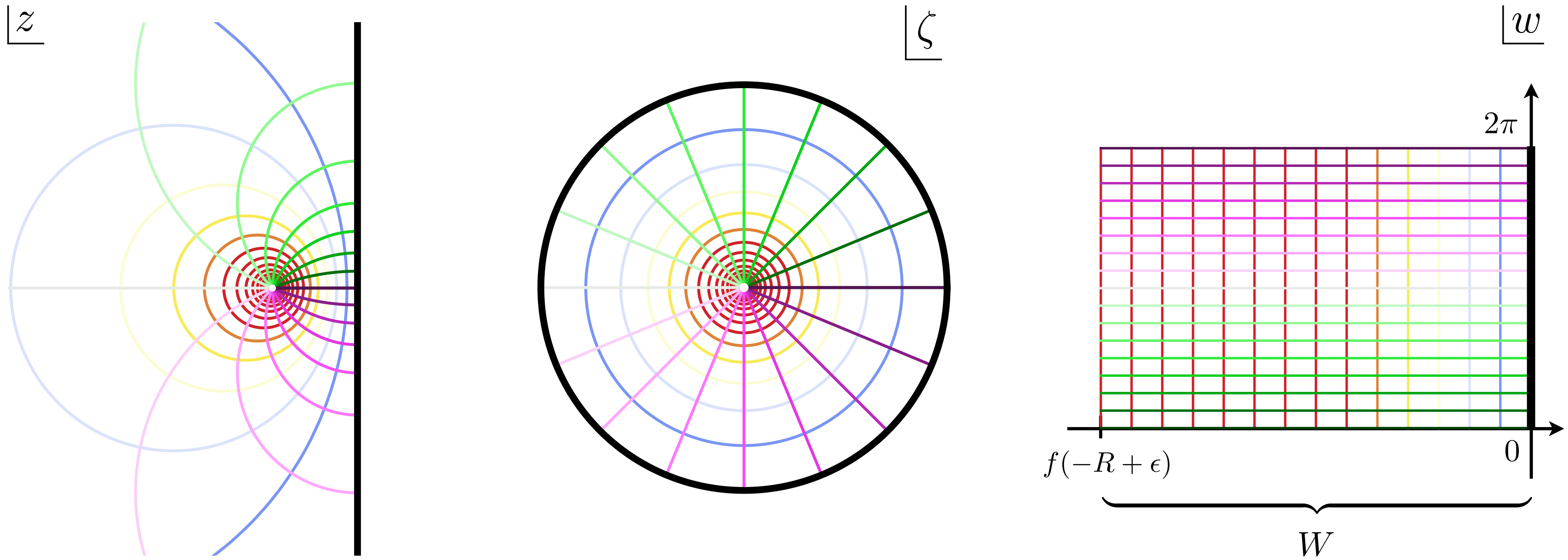
$$K_A = \frac{\beta}{\pi}\int_A \frac{\sinh[\pi(R-x)/\beta]\,\sinh[\pi(x+R)/\beta]}{\sinh(2\pi R/\beta)}\,T_{00}(x)\,dx$$

$$S_A^{(n)} = \frac{c}{6}\left(1+\frac{1}{n}\right)\log\left(\frac{\beta}{\pi\epsilon}\sinh(\pi\ell/\beta)\right) + g_a + g_b + \text{corrections}$$

Entanglement hamiltonians: time-independent cases III

[Cardy, E.T., (2016)]

- Finite interval $A = (-R, 0)$ in the half-line $x < 0$



$$S_A^{(n)} = \frac{c}{12}\left(1 + \frac{1}{n}\right)\log(2R/\epsilon) + g_a + g_b + \dots$$

Now we may have $g_a \neq g_b$

→ Affleck-Ludwig boundary entropy [Affleck, Ludwig, (1991)]

EH: harmonic chains (HC) & free fermions chains (FFC)

- Harmonic chain (periodic b.c.)

$$\widehat{H} = \sum_{i=0}^{L-1} \left(\frac{1}{2m}\, \hat{p}_i^2 + \frac{m\omega^2}{2}\, \hat{q}_i^2 + \frac{\kappa}{2} (\hat{q}_{i+1} - \hat{q}_i)^2 \right) \qquad \hat{\boldsymbol{r}} = \begin{pmatrix} \boldsymbol{q} \\ \boldsymbol{p} \end{pmatrix}$$

$$\widehat{K}_A = \frac{1}{2}\, \hat{\boldsymbol{r}}^{\mathrm{t}} H_A\, \hat{\boldsymbol{r}} \qquad H_A \equiv \begin{pmatrix} M & E \\ E^{\mathrm{t}} & N \end{pmatrix}$$

$$\gamma_A \equiv \begin{pmatrix} Q & R \\ R^{\mathrm{t}} & P \end{pmatrix}$$

Gaussianity and Wick theorem allow to write the EH matrix H_A in terms of the reduced covariance matrix $\gamma_A \equiv \mathrm{Re}\langle \hat{\boldsymbol{r}}\, \hat{\boldsymbol{r}}^{\mathrm{t}} \rangle |_A$

[Peschel, (2003)] [Casini, Huerta, (2009)] [Banchi, Braunstein, Pirandola, (2015)]

- Chain of free fermions

$$\widehat{H} = -\frac{1}{2} \sum_{n=-\infty}^{+\infty} \left(\hat{c}_n^{\dagger}\, \hat{c}_{n+1} + \hat{c}_{n+1}^{\dagger}\, \hat{c}_n \right)$$

$$\widehat{K}_A = \sum_{i,j=1}^{\ell} T_{i,j}\, \hat{c}_i^{\dagger}\, \hat{c}_j \qquad T^{\mathrm{t}} = \log(C_A^{-1} - \mathbf{1})$$

The EH matrix T can be written in terms of the correlation matrix C_A restricted to the subsystem A [Peschel, (2003)]

EH of an interval in the FFC: continuum limit

$$\widehat{K}_A = \sum_{i,j=1}^{N} T_{i,j}\, c_i^\dagger c_j \quad \xrightarrow{\text{continuum limit}} \quad \widehat{K}_A = 2\pi\ell \int_0^\ell \beta(x)\, T_{00}(x)\, dx$$

For an interval A : $\beta(x) = \frac{x}{\ell}\left(1 - \frac{x}{\ell}\right)$

For the Dirac fermion :

$$T_{00}(x) = \frac{1}{2}\left[\, \psi_{\rm R}^\dagger(x)\,(-{\rm i}\,\partial_x)\psi_{\rm R}(x) - \psi_{\rm L}^\dagger(x)\,(-{\rm i}\,\partial_x)\psi_{\rm L}(x) + \text{h.c.}\,\right]$$

- Analysis based on discretisation of deltas [Arias, Blanco, Casini, Huerta, (2016)]

- Standard procedure for the continuum limit (long range hopping)
  $x = i\, s \qquad \ell = N s = \text{const}$ [Eisler, E.T., Peschel, (2019)]

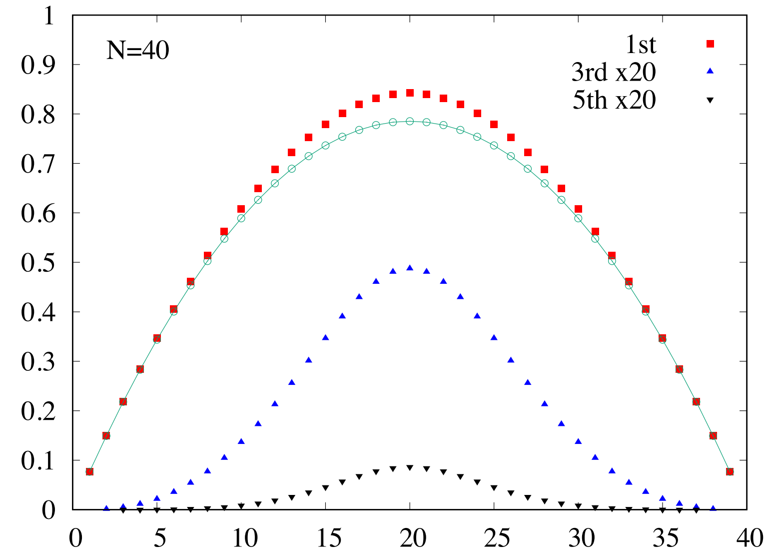
  - Non trivial sums involving the hypergeometric functions must be performed

  - Analytic expressions for the higher derivatives (subleading) terms have been obtained

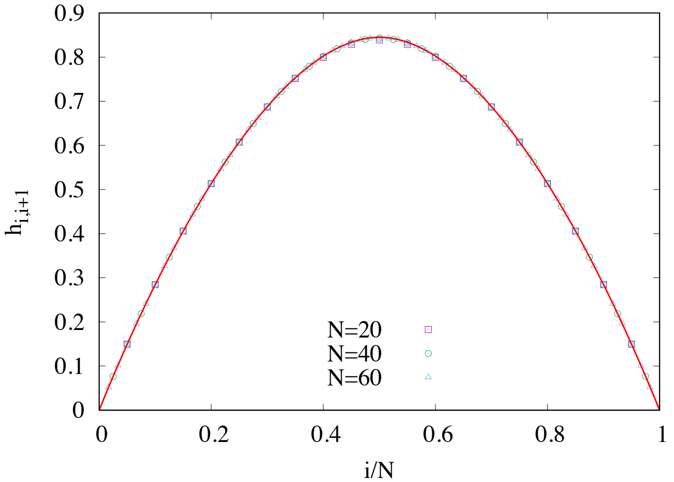
EH of an interval in the FFC

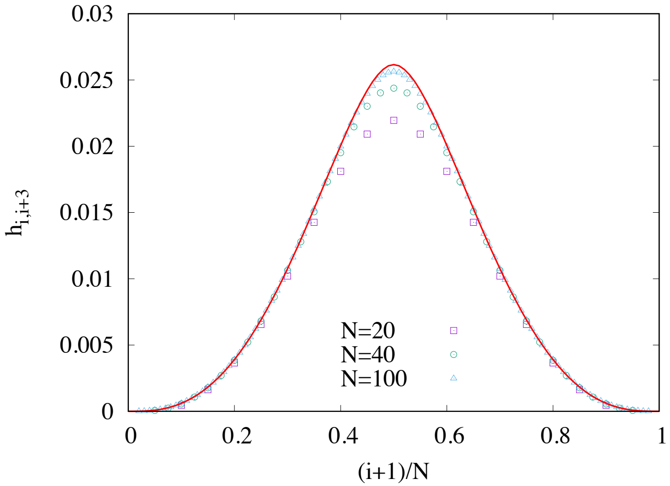
[Eisler, Peschel, (2017)]

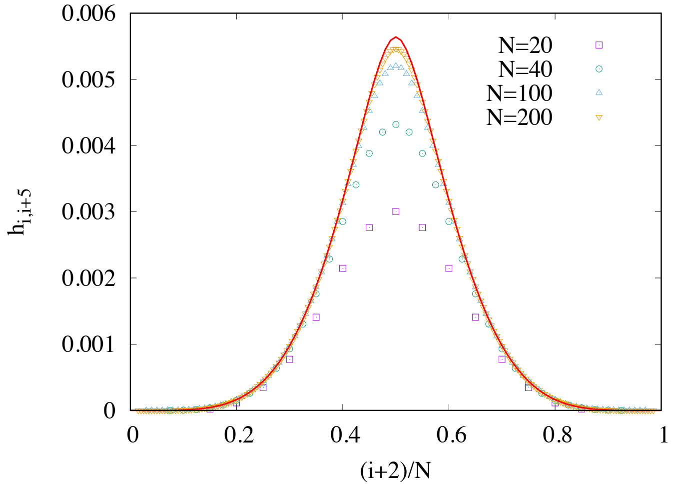
- Free fermions chain: analytic results for the interval A when the number of its sites diverges



$$-h_{i,j} \equiv \lim_{N\to\infty} \frac{T_{i,j}}{N}$$







- The result is written in terms of hypergeometric functions

$$h_{i,i+p} = \pi \, \frac{(2p-1)!!\,(4p-1)!!}{2^p\, p!\,(2p+1)!}\; z^{2p+1}\; {}_3F_2\left(p+\frac{1}{4}, p+\frac{1}{4}, p+\frac{3}{4}\,; p+1, 2(p+1)\,; (4z)^2\right) \qquad z \equiv \frac{i+p}{N}\left(1-\frac{i+p}{N}\right)$$

EH of an interval in the FFC: higher derivatives terms

[Eisler, E.T., Peschel, (2019)]

$$\widehat{K}_A = \sum_{m=0}^{\infty} \frac{1}{m!} \int_0^{\ell} \Big\{ F_m^{(+)}(x)\, \mathcal{H}_+^{(m)}(x) + F_m^{(-)}(x)\, \mathcal{H}_-^{(m)}(x) \Big\}\, dx$$

- Operators:

$$\mathcal{H}_{\pm}^{(m)} \equiv \sum_{k=0}^{m} \binom{m}{k} \left(-\frac{1}{2}\,\partial_x\right)^{m-k} \Psi_{\pm}^{(k)}$$

$$\Psi_{+}^{(k)} \equiv -\frac{1}{2}\left(\psi_{\mathrm{R}}^{\dagger}\,\psi_{\mathrm{R}}^{(k)} + \psi_{\mathrm{L}}^{\dagger}\,\psi_{\mathrm{L}}^{(k)}\right) + \mathrm{h.c.} \qquad \Psi_{-}^{(k)} \equiv -\frac{\mathrm{i}}{2}\left(\psi_{\mathrm{R}}^{\dagger}\,\psi_{\mathrm{R}}^{(k)} - \psi_{\mathrm{L}}^{\dagger}\,\psi_{\mathrm{L}}^{(k)}\right) + \mathrm{h.c.}$$

- Weight functions:

$$\begin{cases} F_m^{(+)}(x) \;\equiv\; \delta_{m,0}\, t_0(x) + 2\, s^m \displaystyle\sum_{r=1}^{\infty} r^m \cos(r q_F s)\, t_r(x) \\[2ex] F_m^{(-)}(x) \;\equiv\; 2\, s^m \displaystyle\sum_{r=1}^{\infty} r^m \sin(r q_F s)\, t_r(x) \end{cases}$$

- $m = 0$ $\begin{cases} \mathcal{H}_-^{(0)} = 0 \\ F_0^{(+)}(x) = 0 \end{cases}$ $\qquad m = 1$ $\begin{cases} \mathcal{H}_+^{(1)} = 0 \\ \mathcal{H}_-^{(1)} = T_{00} \qquad F_1^{(-)}(x) = 2\pi\ell\,\beta(x) \end{cases}$

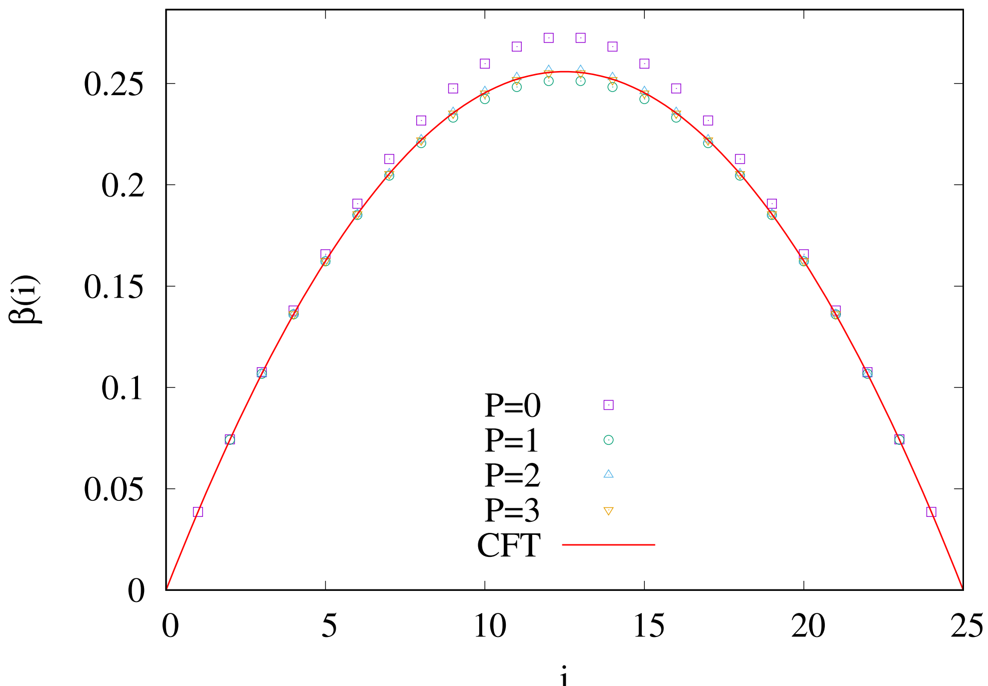
EH of an interval in the FFC: finite size & finite T

- At finite size and finite T, we observe numerically that

$$\beta(i) = \frac{1}{\pi N} \sum_{p=0}^{P} (-1)^p (2p+1) H_{i-p,i+p+1}$$

- Periodic chain of length L

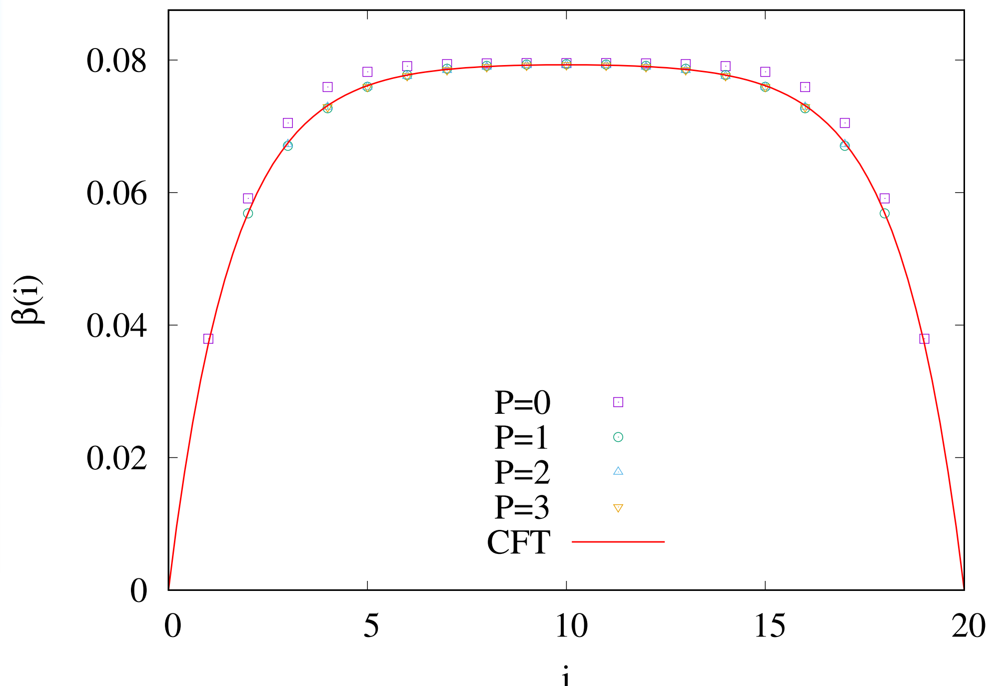
$$\beta(x) = \frac{L}{\pi\,\ell}\,\frac{\sin(\pi\,x/L)\sin(\pi(\ell-x)/L)}{\sin(\pi\,\ell/L)}$$



$N = 25$ $\qquad \ell/L = 1/6$

- Infinite chain at temperature $1/\beta$

$$\beta(x) = \frac{\beta}{\pi\,\ell}\,\frac{\sinh(\pi\,x/\beta)\sinh(\pi(\ell-x)/\beta)}{\sinh(\pi\,\ell/\beta)}$$



$N = 20$ $\qquad \beta = 10$

EH of an interval in the HC

- Decomposition of the EH:

$$\widehat{K}_A = \frac{\widehat{H}_M + \widehat{H}_N}{2} \qquad \widehat{H}_M \equiv \sum_{i,j=1}^{L} M_{i,j}\, \hat{q}_i\, \hat{q}_j \qquad \widehat{H}_N \equiv \sum_{i,j=1}^{L} N_{i,j}\, \hat{p}_i\, \hat{p}_j$$

- The correlation matrices restricted to A determine the matrices M and N

[Peschel, (2003)] [Casini, Huerta, (2009)]

$$H_A = M \oplus N \equiv \Big(h\big(\sqrt{P_A Q_A}\,\big) \oplus h\big(\sqrt{Q_A P_A}\,\big)\Big)\big(P_A \oplus Q_A\big)$$

$$h(y) \equiv \frac{1}{y}\,\log\!\left(\frac{y+1/2}{y-1/2}\right)$$

- Scaling of the diagonals of M and N

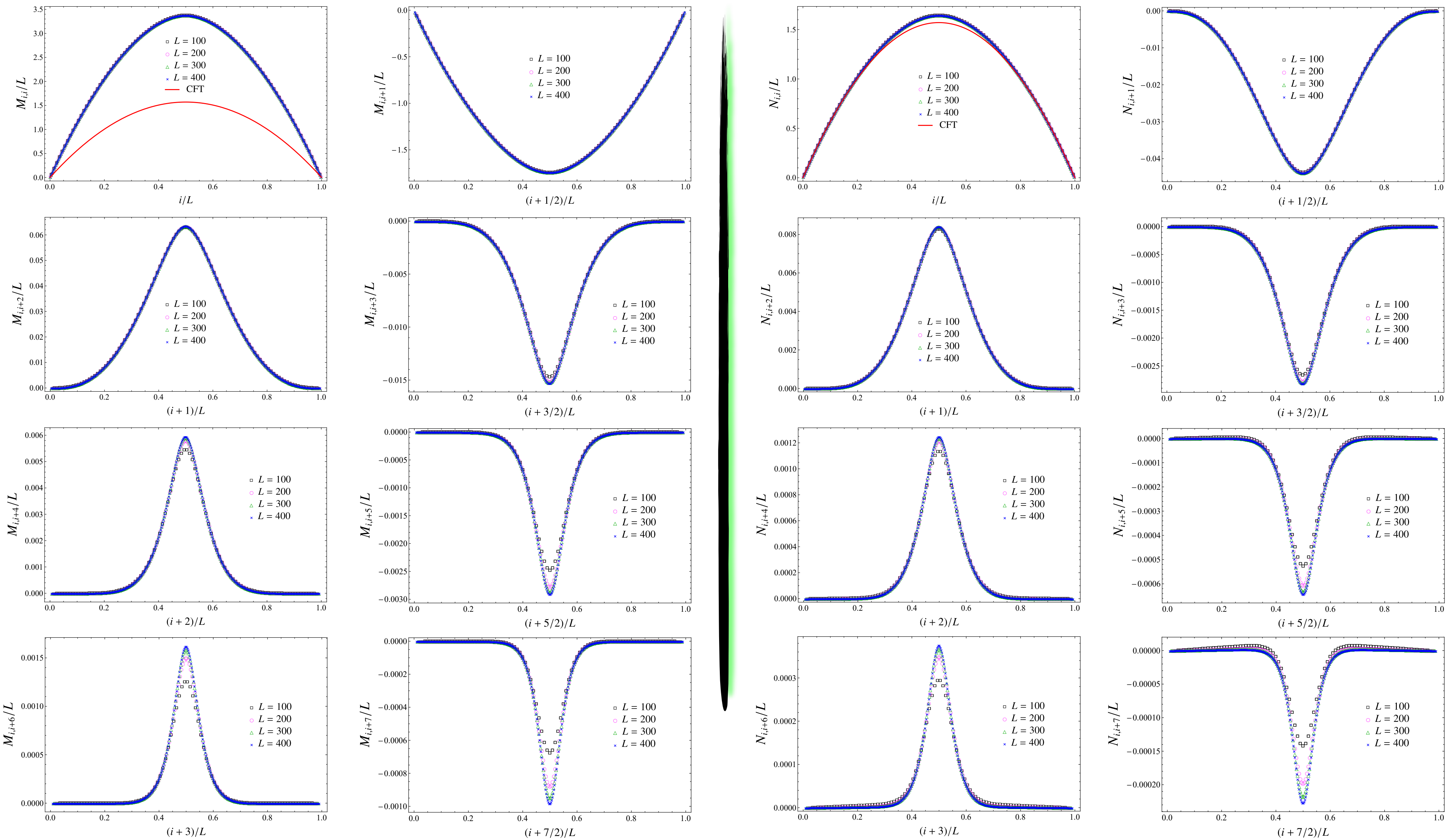
[Di Giulio, E.T., (2019)]

$$\lim_{L\to\infty} \frac{M_{i,i+k}}{L} \equiv \mu_k(x_k) \qquad \lim_{L\to\infty} \frac{N_{i,i+k}}{L} \equiv \nu_k(x_k) \qquad x_k \equiv \frac{1}{L}\left(i + \frac{k}{2}\right)$$

EH of an interval in the HC: diagonals of M and N

[Di Giulio, E.T., (2019)]

Numerical results indicate that $\mu_k(x_k)$ and $\nu_k(x_k)$ are well defined



EH of an interval in the HC: continuum limit

- The fields $\Phi(x)$ and $\Pi(x)$ are introduced in the standard way

$$\hat{q}_i \longrightarrow \Phi(x)$$
$$\hat{p}_i \longrightarrow a\,\Pi(x)$$

Also $\hat{q}_{i+k} \longrightarrow \Phi(x+ka)$ and $\hat{p}_{i+k} \longrightarrow a\,\Pi(x+ka)$ are needed because long range interactions occur

- The continuum limit leads to introduce particular combinations of diagonals

- $$\mathcal{M}^{(0)}_{k_{\max}}(x) \equiv \lim_{L\to\infty} \frac{\mathsf{M}^{(0)}_{k_{\max}}(i)}{L} = \mu_0(x) + 2\sum_{k=1}^{k_{\max}} \mu_k(x) \qquad \mathsf{M}^{(0)}_{k_{\max}}(i) \equiv M_{i,i} + 2\sum_{k=1}^{k_{\max}} M_{i,i+k}$$

- $$\mathcal{N}^{(0)}_{k_{\max}}(x) \equiv \lim_{L\to\infty} \frac{\mathsf{N}^{(0)}_{k_{\max}}(i)}{L} = \nu_0(x) + 2\sum_{k=1}^{k_{\max}} \nu_k(x) \qquad \mathsf{N}^{(0)}_{k_{\max}}(i) \equiv N_{i,i} + 2\sum_{k=1}^{k_{\max}} N_{i,i+k}$$

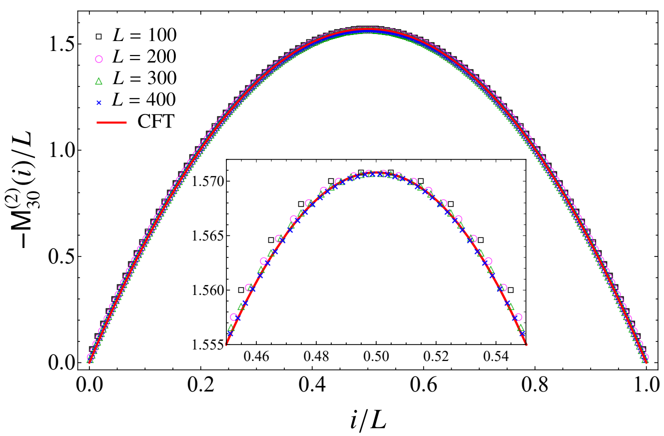
- $$\mathcal{M}^{(2)}_{k_{\max}}(x) \equiv \lim_{L\to\infty} \frac{\mathsf{M}^{(2)}_{k_{\max}}(i)}{L} \equiv \sum_{k=1}^{k_{\max}} k^2 \mu_k(x_k) \qquad \mathsf{M}^{(2)}_{k_{\max}}(i) \equiv \sum_{k=1}^{k_{\max}} k^2 M_{i,i+k}$$

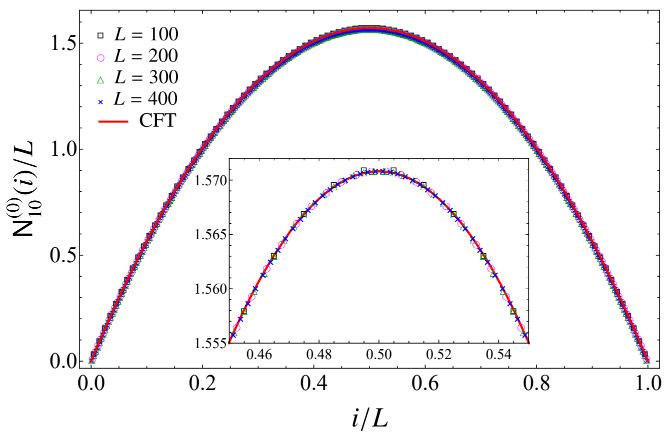
- $$\mathcal{M}^{(2)}_{2,k_{\max}}(x) \equiv \lim_{L\to\infty} \frac{\mathsf{M}^{(2)}_{2,k_{\max}}(i)}{L} \equiv \sum_{k=1}^{k_{\max}} k^2 \mu_{2,k}(x_k) \qquad \mathsf{M}^{(2)}_{2,k_{\max}}(i) \equiv \sum_{k=1}^{k_{\max}} k^2 \big(M_{i+1,i+1+k} - 2M_{i,i+k} + M_{i-1,i-1+k}\big)$$

EH of an interval in the massless HC: continuum limit

$$\frac{H_M + H_N}{2} = \frac{\ell}{a^2}\int_0^{\ell} \frac{1}{2}\left[\mathcal{M}^{(0)}_{\infty}(x) + \frac{1}{4}\,\mathcal{M}^{(2)}_{2,\infty}(x)\right]\Phi(x)^2\,dx$$

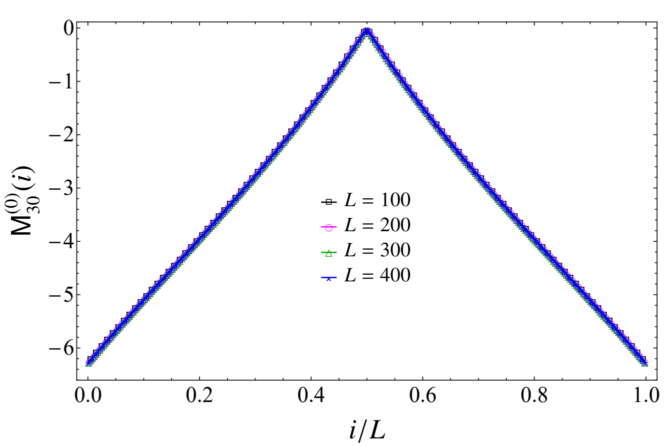
$$+\,\ell\int_0^{\ell}\frac{1}{2}\left[\mathcal{N}^{(0)}_{\infty}(x)\,\Pi(x)^2 - \mathcal{M}^{(2)}_{\infty}(x)\big(\Phi'(x)\big)^2\right]dx + O(a)$$

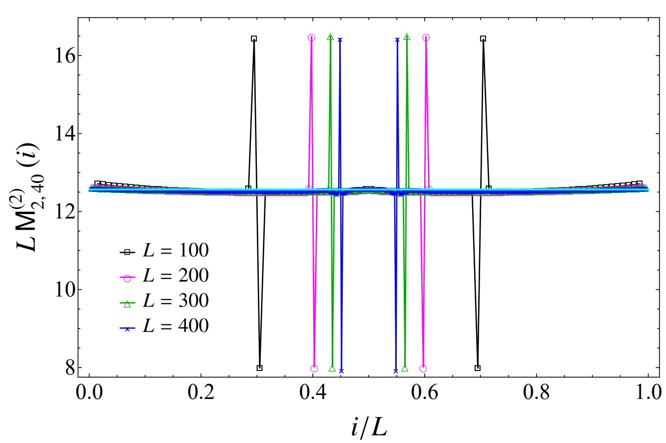




$$-\,\mathcal{M}^{(2)}_{\infty}(x) = \mathcal{N}^{(0)}_{\infty}(x) = \beta(x)$$

$$\beta(x) \equiv 2\pi\,\frac{x}{\ell}\left(1 - \frac{x}{\ell}\right)$$



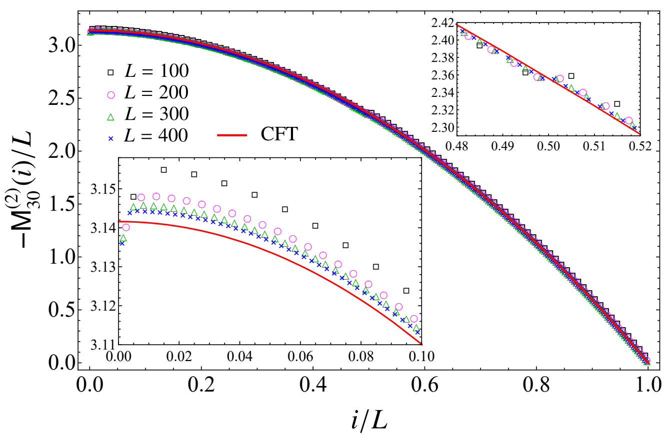


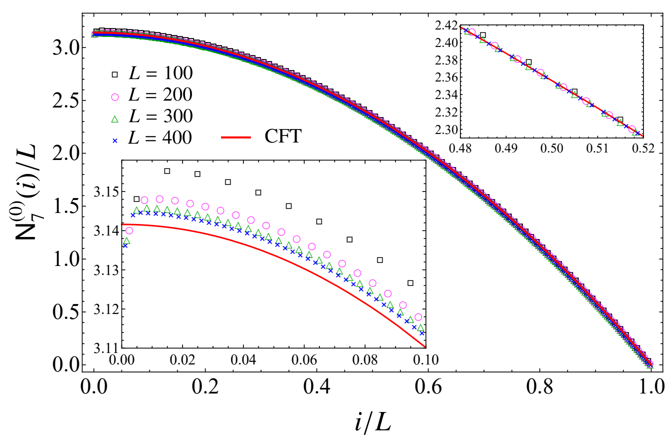
$$\mathcal{M}^{(0)}_{\infty}(x) = \mathcal{M}^{(2)}_{2,\infty}(x) = 0$$

EH of an interval at the beginning of the half line

$$\frac{H_M + H_N}{2} = \frac{\ell}{a^2} \int_0^{\ell} \frac{1}{2} \left[\mathcal{M}_{\infty}^{(0)}(x) + \frac{1}{4}\, \mathcal{M}_{2,\infty}^{(2)}(x) \right] \Phi(x)^2 \, dx$$

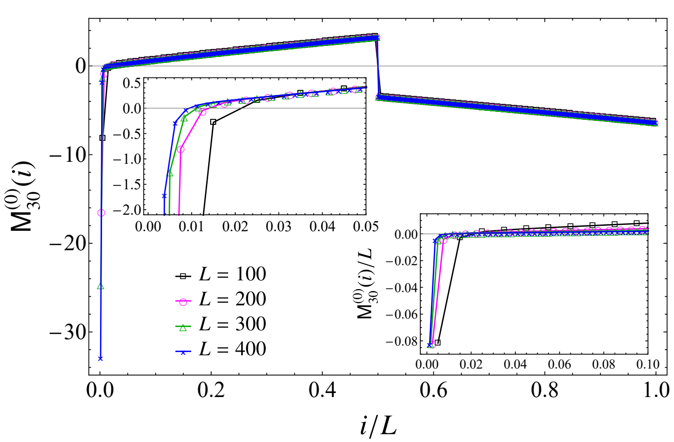
$$+ \ell \int_0^{\ell} \frac{1}{2} \left[\mathcal{N}_{\infty}^{(0)}(x)\, \Pi(x)^2 + \mathcal{M}_{1,\infty}^{(2)}(x)\, \Phi'(x)\, \Phi(x) + \mathcal{M}_{\infty}^{(2)}(x)\, \Phi''(x)\, \Phi(x) \right] dx$$

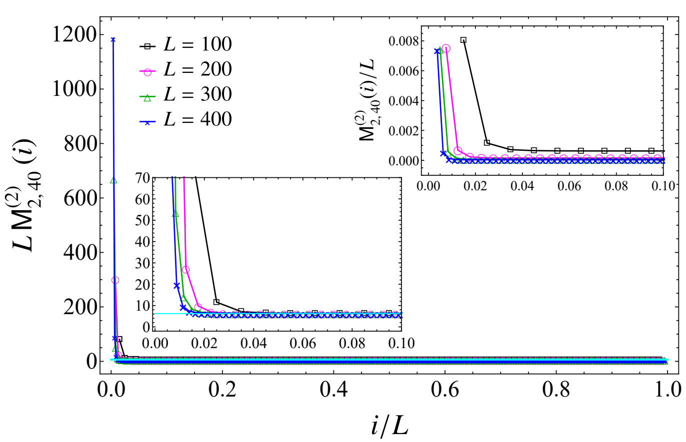


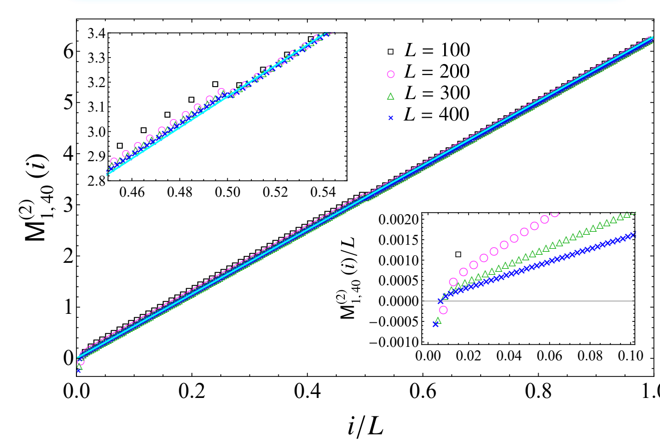


$$-\mathcal{M}_{\infty}^{(2)}(x) = \mathcal{N}_{\infty}^{(0)}(x) = \beta(x)$$

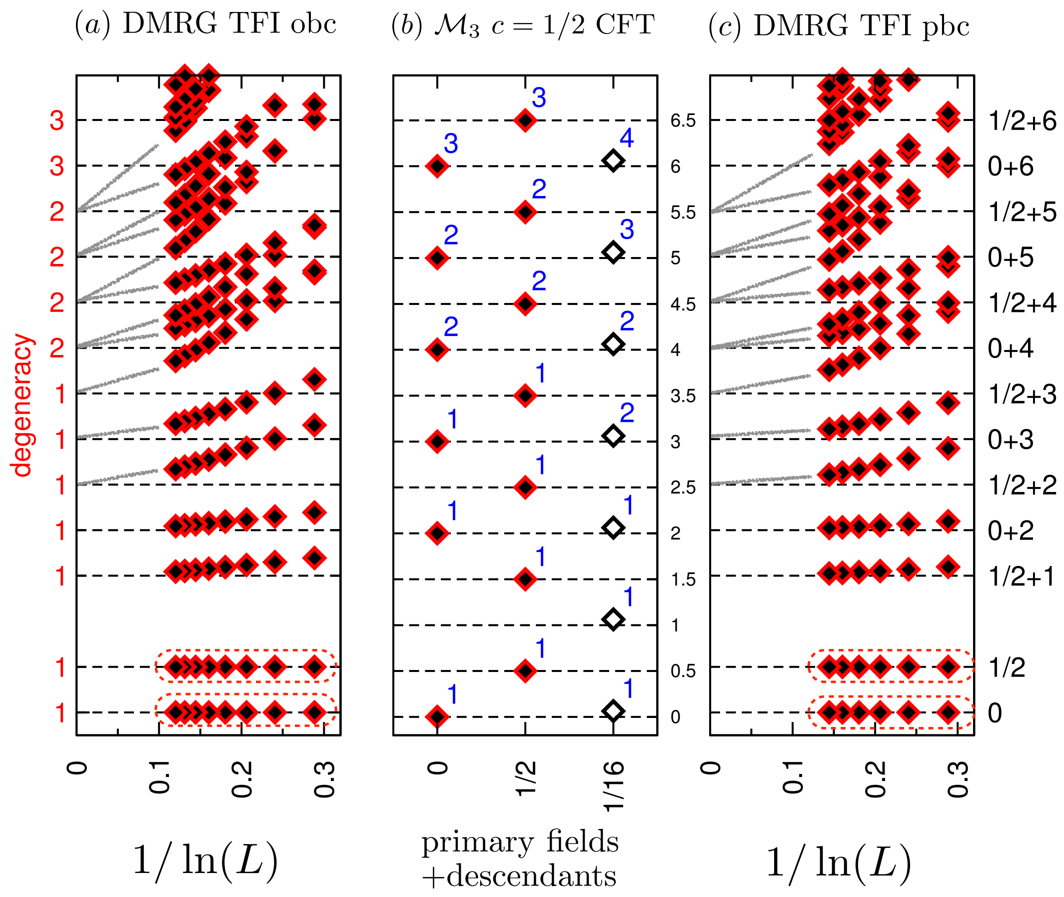
$$\mathcal{M}_{\infty}^{(0)}(x) = \mathcal{M}_{1,\infty}^{(2)}(x) = \\ = \mathcal{M}_{2,\infty}^{(2)}(x) = 0$$







Entanglement spectra for an interval: Ising chain



- Correspondence between entanglement spectra and conformal spectra : Numerical evidences in some spin chains [Läuchli, (2013)]

→ Two Virasoro towers $0 \otimes \frac{1}{2}$

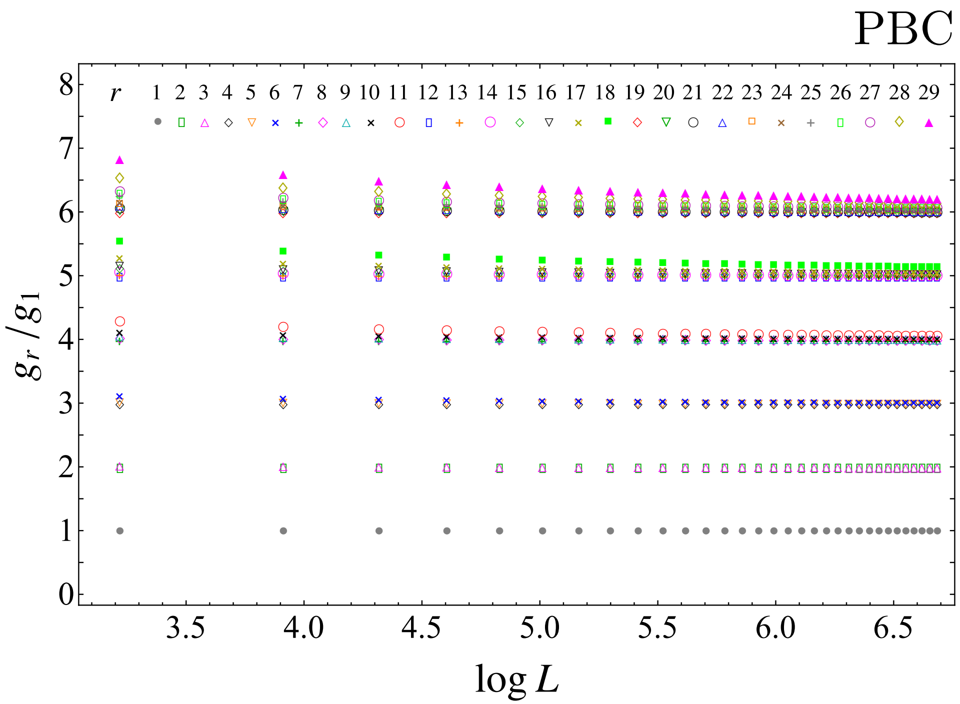
→ Match with the operator content of the Ising CFT on the annulus with free boundary conditions on both sides [Cardy, (1989)]

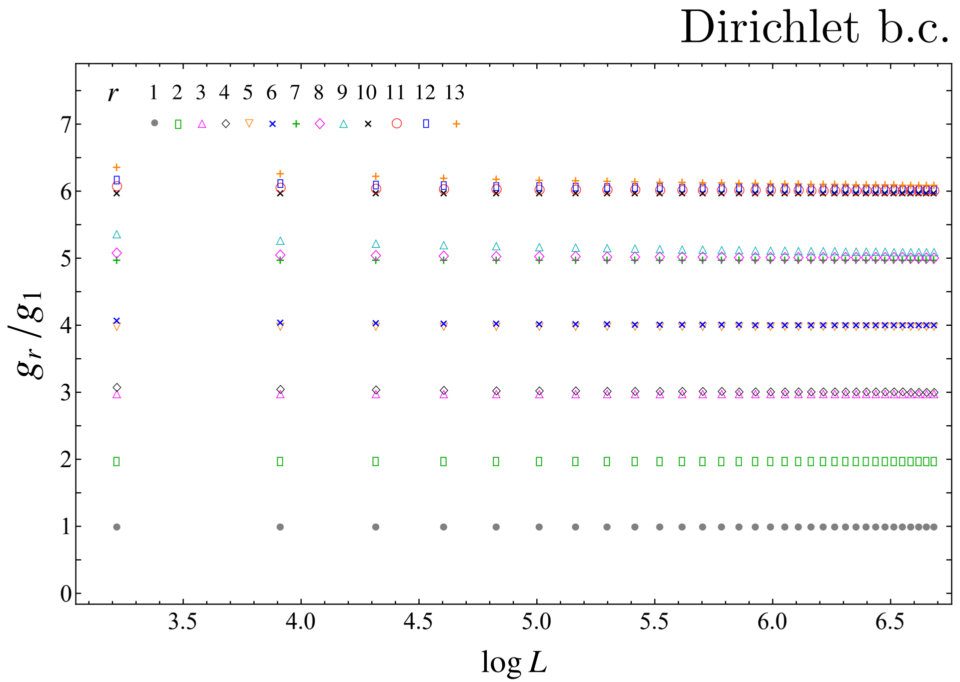
- Agreement with the BCFT approach (mapping to the annulus) [Cardy, E.T., (2016)]

Entanglement spectra for an interval: massless HC

[Di Giulio, E.T., (2019)]

- Harmonic chains





- Agreement with the CFT prediction
  [Cardy, E.T., (2016)]

$$\frac{g_r}{g_1} = \frac{\Delta_r}{\Delta_1}$$

- Data compatible with Neumann b.c. imposed at the boundary introduced around the entangling points by the regularisation procedure

(Global) Quantum quenches

[Calabrese, Cardy, (2005), (2007)]

- Quantum quench:
  - System prepared in the ground state $|\psi_0\rangle$ of H_0
  - At $t = 0$ a sudden change is performed

Unitary evolution: $|\psi(t)\rangle = e^{-\mathrm{i}Ht}|\psi_0\rangle$ $\quad \rho(t) = |\psi(t)\rangle\langle\psi(t)|$

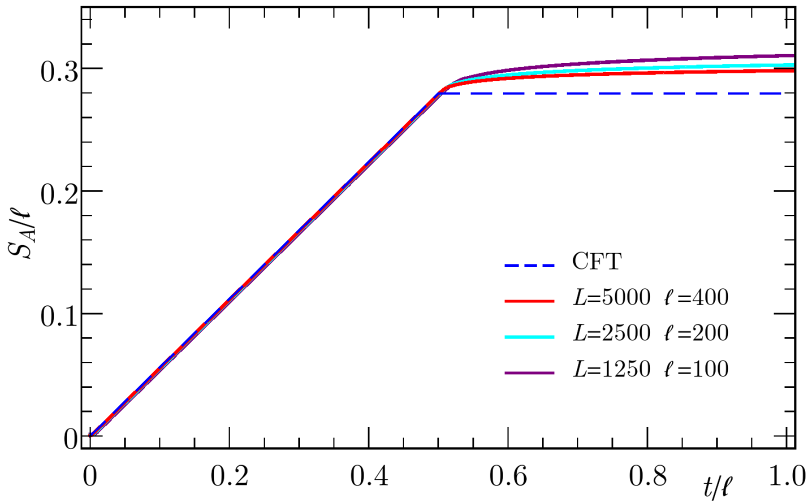
Global quench

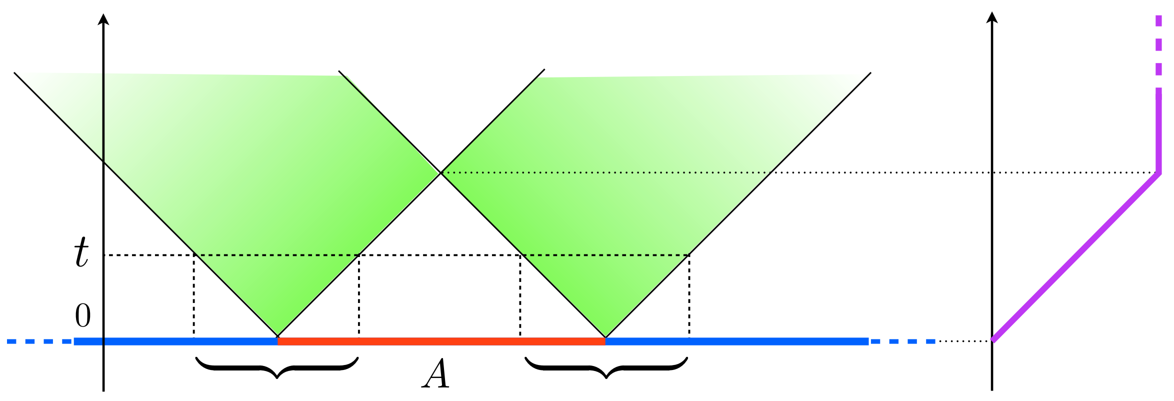
A parameter of H_0 is modified
(e.g. mass switched off)

Local quench

Interaction modified in one point
(e.g. two half-lines joined)

→ $S_A(t)$ when H is the Hamiltonian of a CFT $\quad S_A \simeq 2\pi c\, t/(3\tau_0) \quad t/\ell < 1/2$





Figures from [Coser, E.T., Calabrese, (2014)]

EH of a semi-infinite line after a global quench in CFT

[Cardy, E.T., (2016)]

- Conformal symmetry and a proper analytic continuation provides $\widehat{K}_A$ when A is a semi-infinite line after a global quench

$$\widehat{K}_A = \frac{\tau_0}{\pi} \int_{-t}^{\infty} \frac{\sinh(\pi[x+t]/\tau_0)\, \cosh(\pi[x-t]/\tau_0)}{\cosh(2\pi t/\tau_0)}\, T(x)\, dx + \frac{\tau_0}{\pi} \int_{t}^{\infty} \frac{\sinh(\pi[x-t]/\tau_0)\, \cosh(\pi[x+t]/\tau_0)}{\cosh(2\pi t/\tau_0)}\, \overline{T}(x)\, dx$$

  - The expected linear growth of S_A is recovered.

- Gaps in the entanglement spectrum

$$g_{a,0} \simeq \frac{\pi \tau_0\, \Delta_a}{2\, t}$$

- This $\widehat{K}_A$ provides a natural candidate for the contour function $s_A(x,t)$

Entanglement spectrum after a global quench: Ising chain

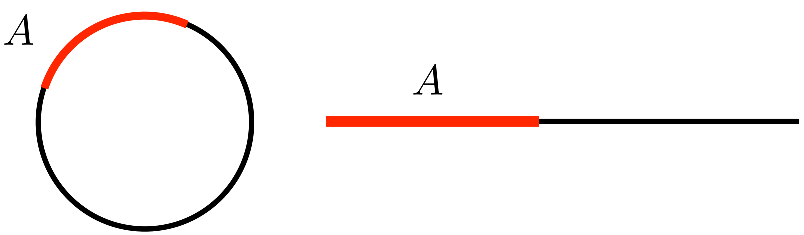
[Surace, Tagliacozzo, E.T., (2019)]

- Transverse Field Ising Chain

$$H(\theta) = -\frac{1}{2}\left(\sum_{i=1}^{L-1}\sigma_i^x\sigma_{i+1}^x + \cot\theta\sum_{i=1}^{L}\sigma_i^z + \eta\,\sigma_L^x\sigma_1^x\right)$$

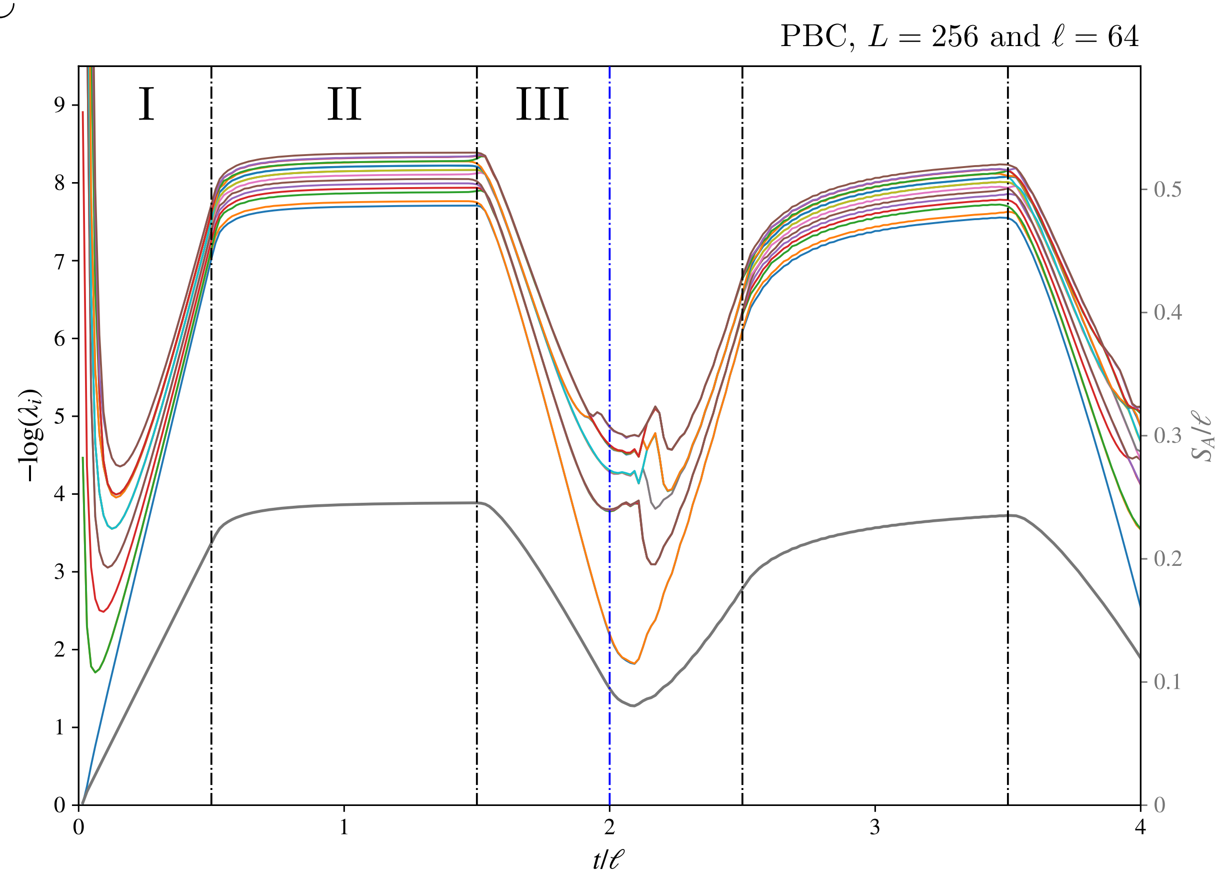
$$\begin{cases} \text{ferromagnetic (ordered) phase } \frac{\pi}{4} < \theta < \frac{\pi}{2} \\ \text{paramagnetic (disordered) phase } 0 < \theta < \frac{\pi}{4} \end{cases}$$

$\eta = 1$ for PBC and $\eta = 0$ for OBC



- Temporal evolution of the entanglement spectrum after a quench at the QCP

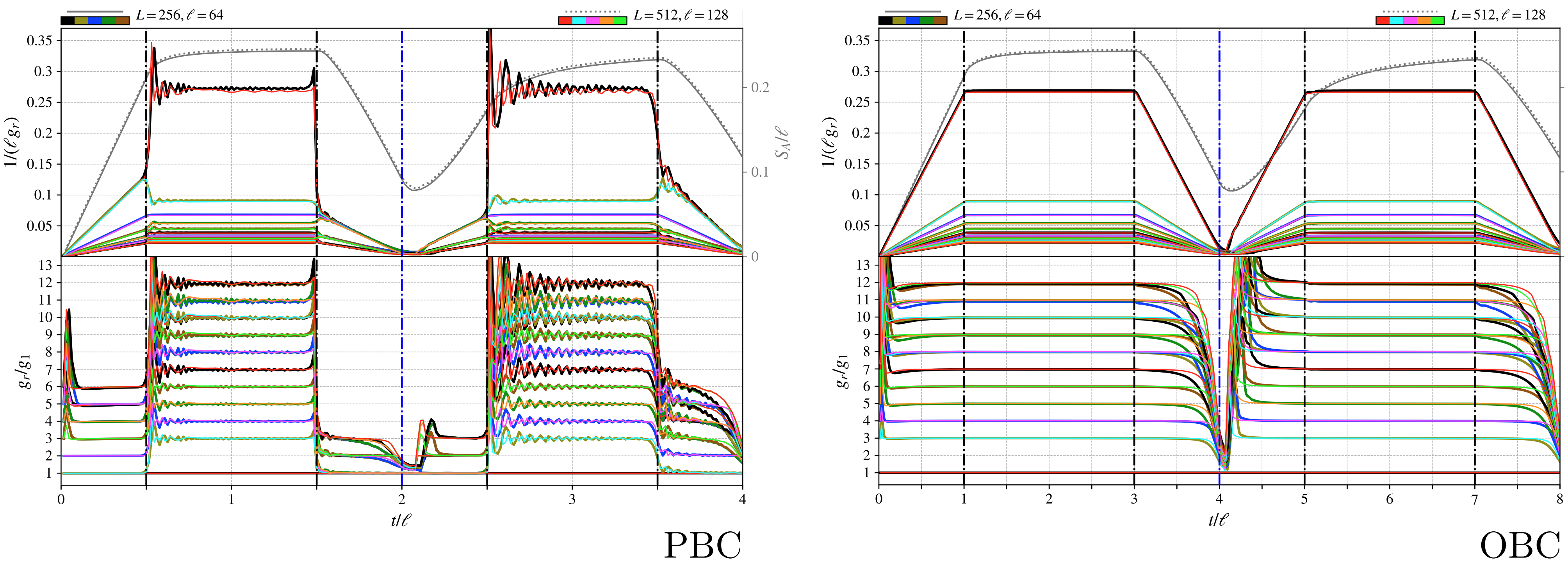
$\theta = \pi/8 \to \theta = \pi/4$



Entanglement spectrum after a global quench: Ising chain

- <u>Quench at the critical point</u> (data for $\theta_0 = \pi/8 \to \theta = \pi/4$)

  Temporal evolution of the gaps of the entanglement spectrum

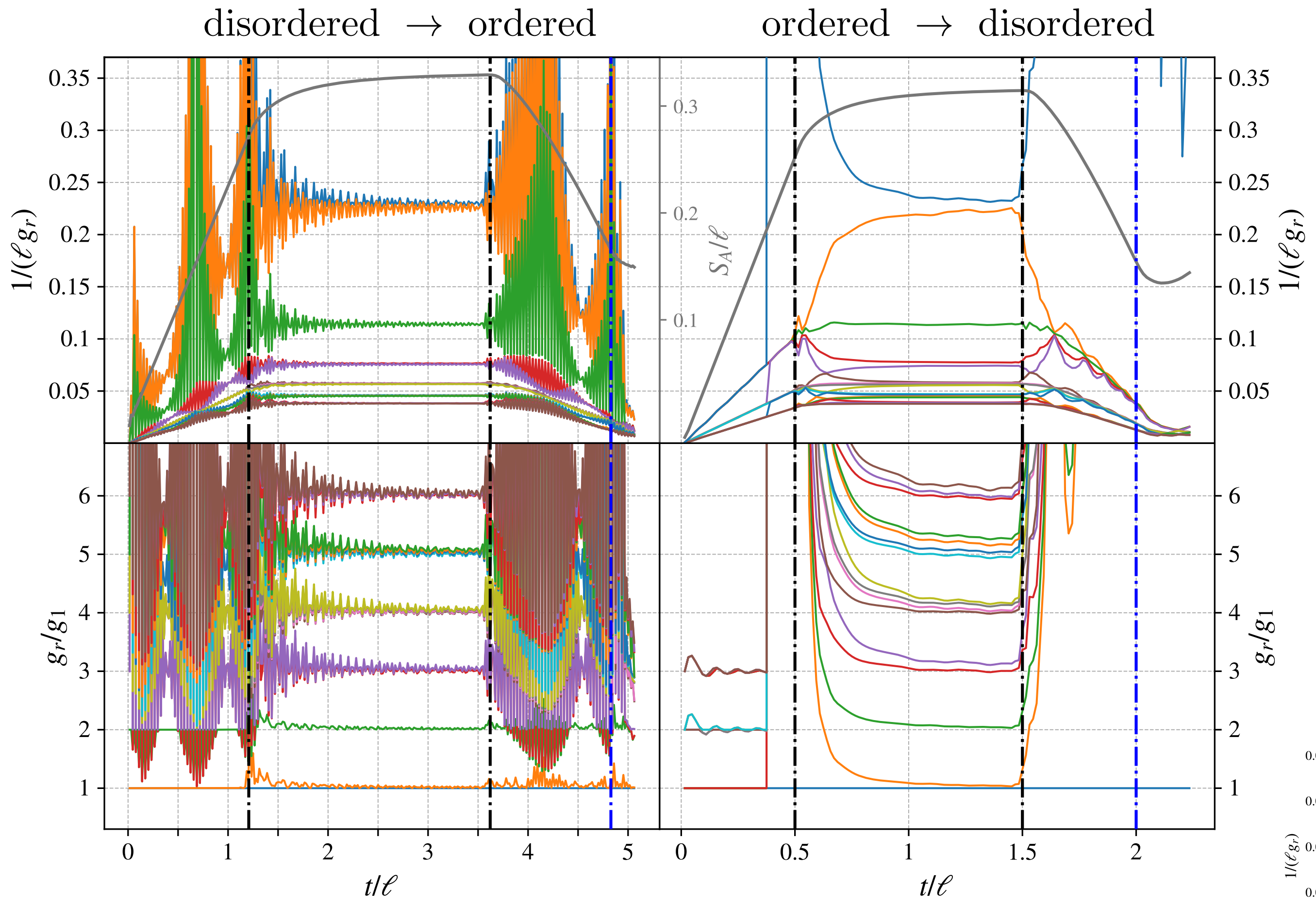


→ The data for the first two regimes can be explained through a BCFT approach

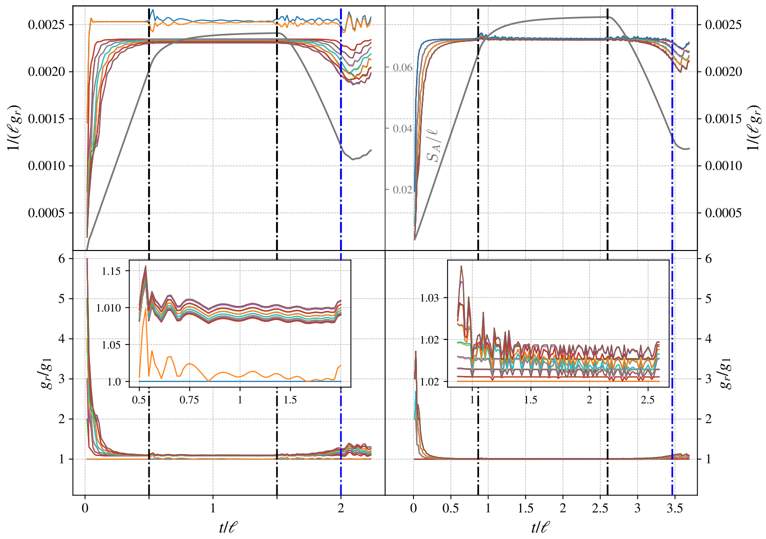
- Data are consistent with free b.c. for a and b_0

Entanglement spectrum after a global quench: Ising chain

- Quenches between different phases (across the QCP)



- Quenches within the same phase



Global quenches protocols in a FFC and in the HC

- Harmonic chain: quench of the frequency parameter $\omega_0 \to \omega$
  [Calabrese, Cardy, (2007)]

  We mainly considered the quench having $\omega_0 = 1$ and $\omega = 0$

- Free fermions chain: [Eisler, Peschel, (2007)]

  the initial state is the ground state of a dimerised chain

$$\widehat{H}_0 = -\frac{1}{2} \sum_{n=-\infty}^{+\infty} t_n \big(\hat{c}_n^\dagger \, \hat{c}_{n+1} + \hat{c}_{n+1}^\dagger \, \hat{c}_n \big) \qquad \begin{cases} t_{2n} = 1 \\ t_{2n+1} = 0 \end{cases}$$

  At $t = 0$ the inhomogeneity is removed

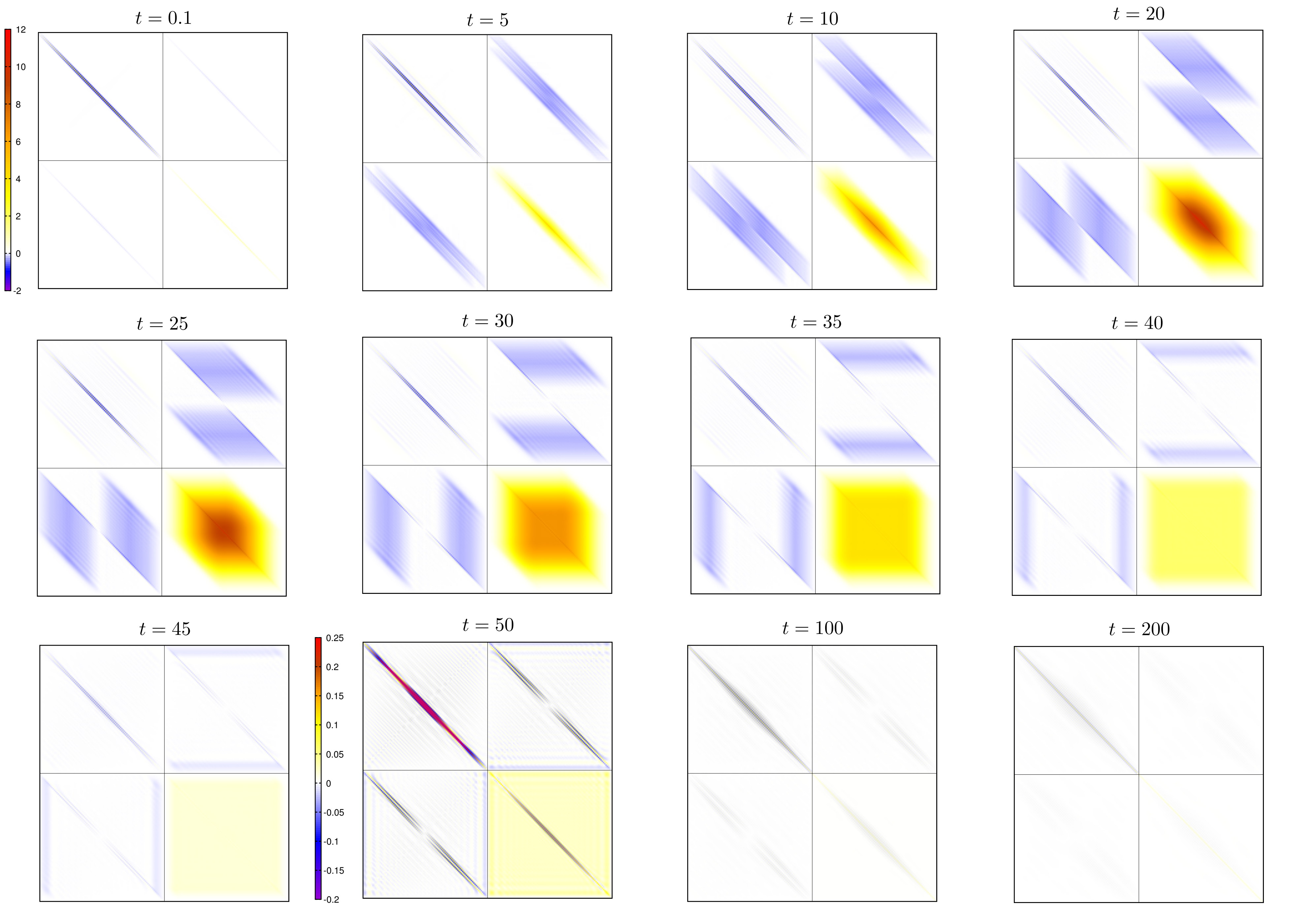
$$\widehat{H} = -\frac{1}{2} \sum_{n=-\infty}^{+\infty} \big(\hat{c}_n^\dagger \, \hat{c}_{n+1} + \hat{c}_{n+1}^\dagger \, \hat{c}_n \big)$$

  - $C_A(t)$ for an interval is known analytically

Evolution of the EH matrix in the HC

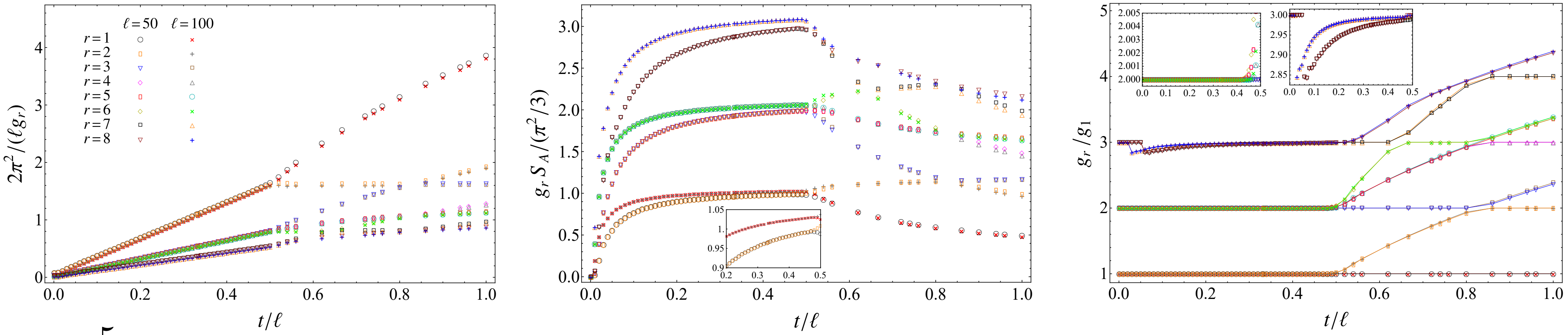
$\ell = 100 \quad \omega_0 = 1 \quad \omega = 0$

[Di Giulio, Arias, E.T., (2019)]

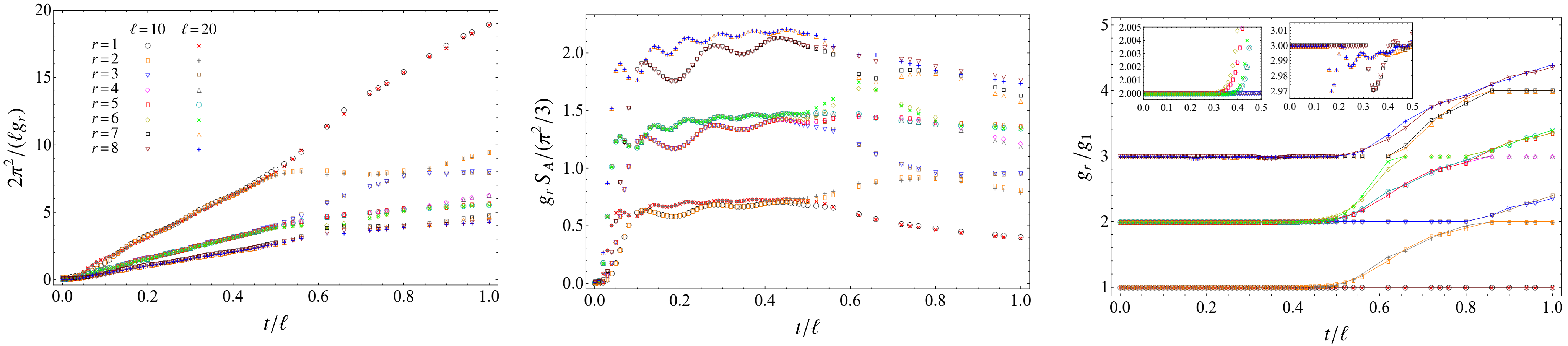


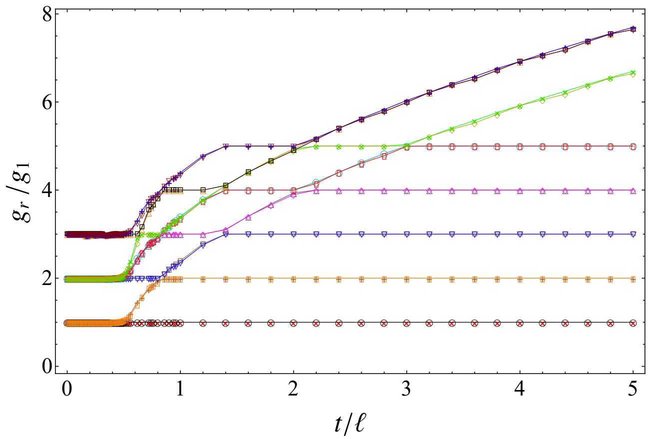
Gaps in the entanglement spectrum in the HC

$\omega_0 = 1$



$\omega_0 = 5$





- Linear growths of $1/g_r$ for $t/\ell < 1/2$ that depend on the CFT spectrum

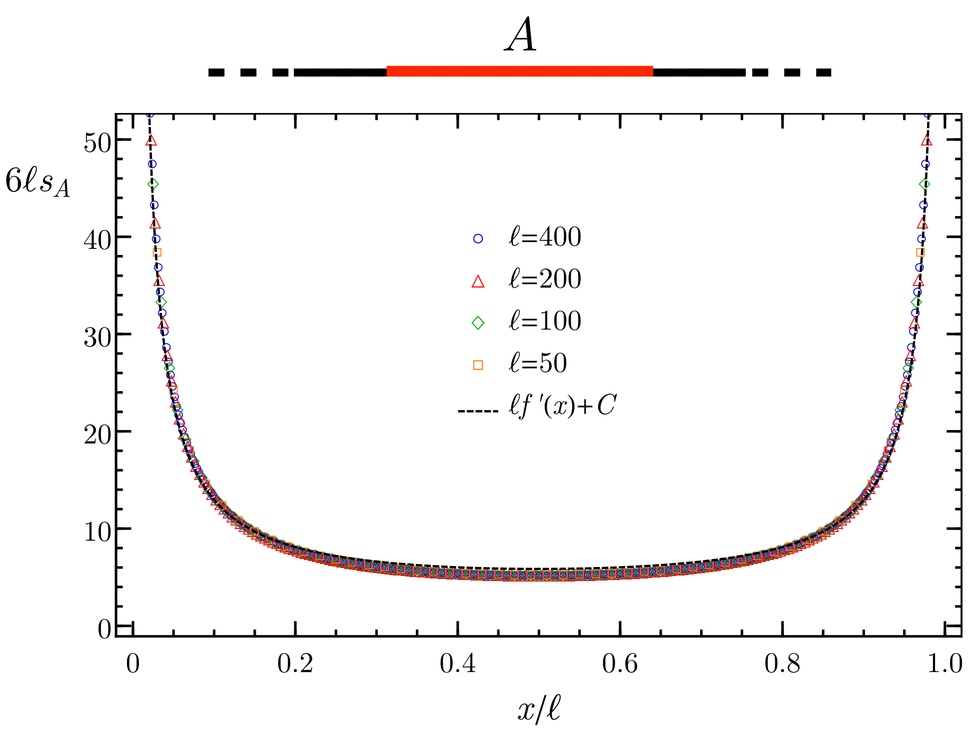
Contour for EE in the HC

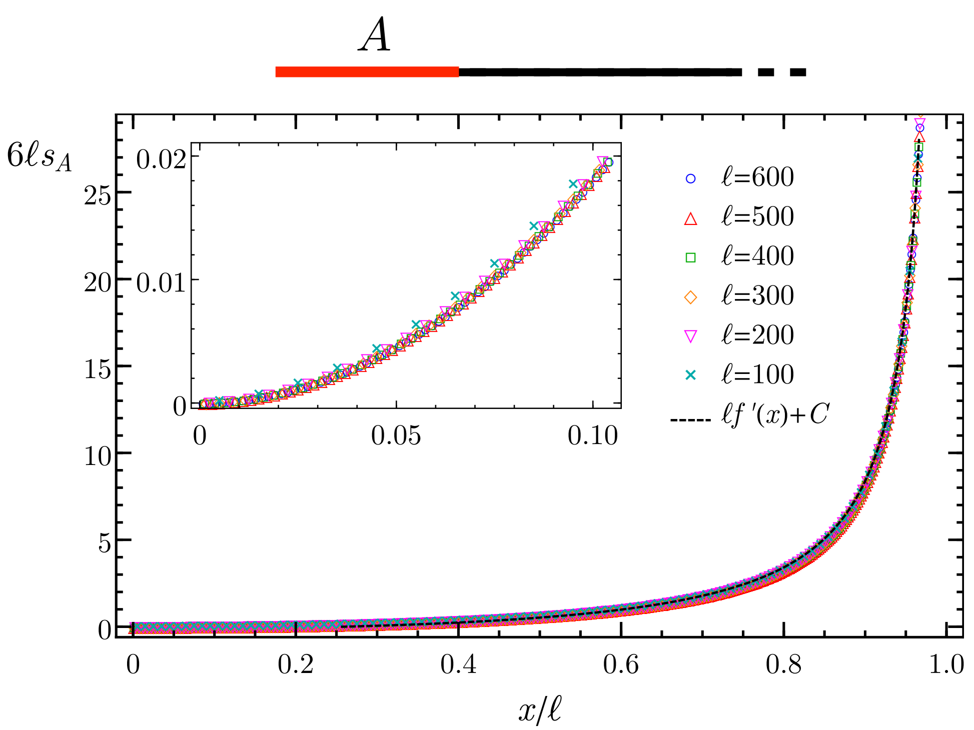
- Spatial structure of entanglement [Botero, Reznik, (2004)] [Chen, Vidal, (2014)]

$$S_A = \sum_{i \in A} s_A(i) \qquad s_A(i) \geqslant 0$$

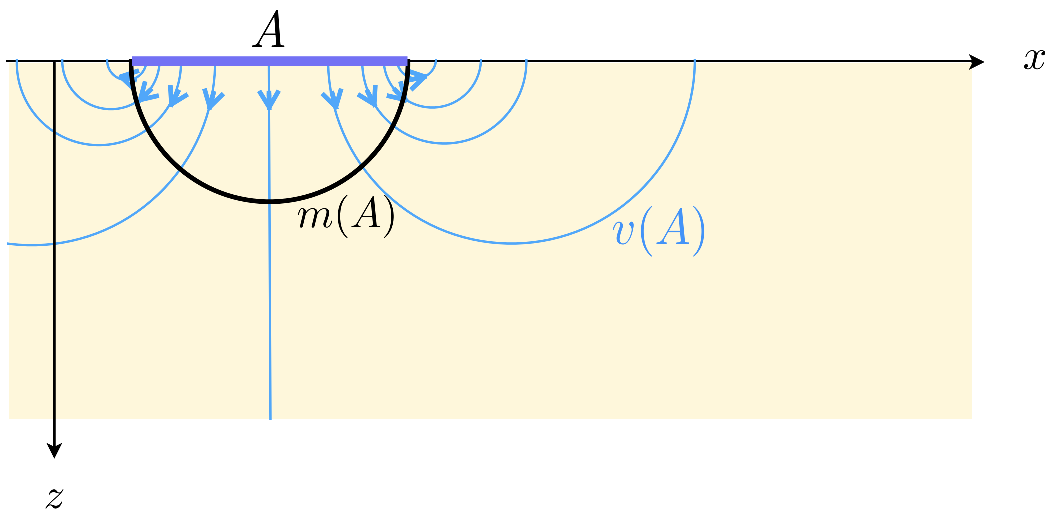
  - Other constraints are imposed on the contour function
  - A list of properties that provides $s_A(i)$ uniquely is not known

- In HC we use $s_A(i)$ constructed in [Coser, De Nobili, E.T., (2017)] that has been compared to the weight function in the EH from CFT





Contours for the holographic entanglement entropy?



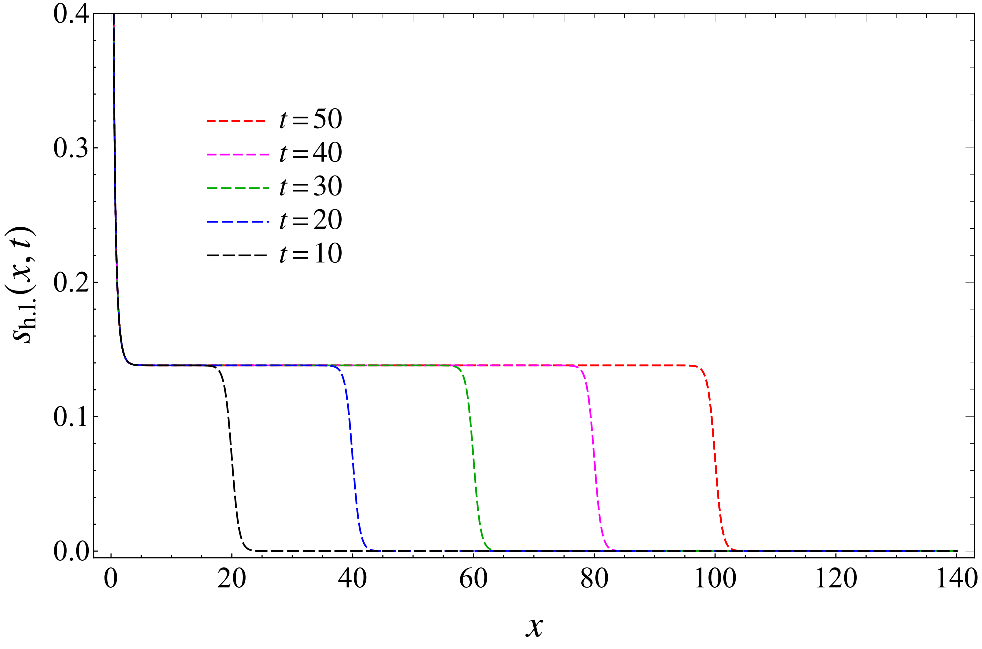
E.g.: AdS$_3$ $\quad \frac{1}{z^2}\left(-\,dt^2 + dz^2 + dx^2 \right)$

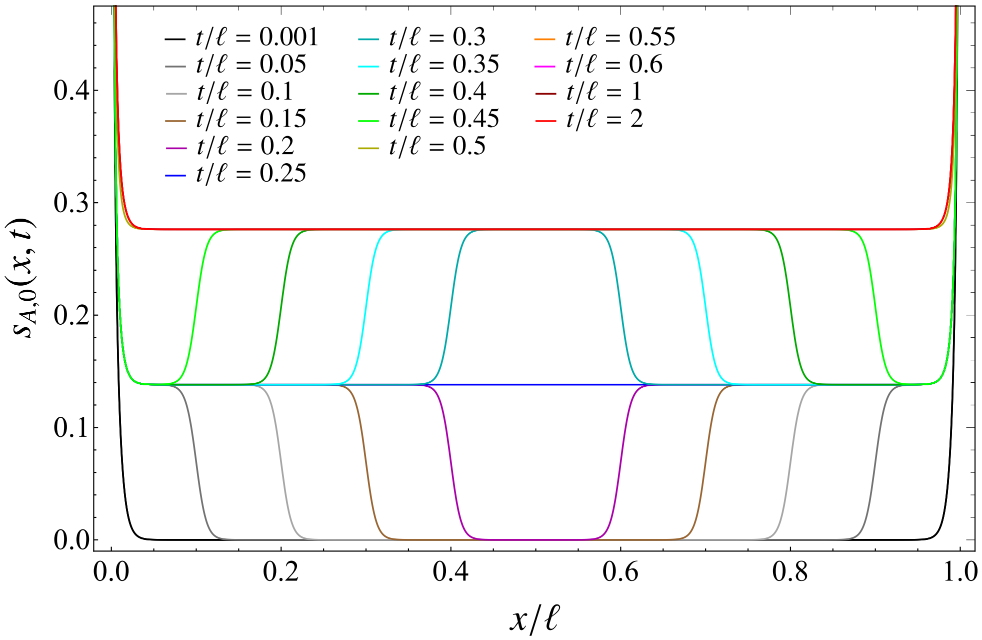
- The Ryu-Takayanagi prescription can be reformulated in terms of flows through A [Headrick, Freedman, (2016)] [Headrick, Hubeny, (2017)]

$$\begin{cases} \nabla_\mu v^\mu = 0 \\ |v| \leqslant \dfrac{1}{4G_{\rm N}} \end{cases} \xrightarrow{\text{max-cut min-flow theorem}} \boxed{S_A = \frac{\min\left[\text{area}(m(A))\right]}{4G_N} = \max \int_A v}$$

- A contour for the holographic entanglement entropy could be identified with a flow maximising the flux. [E.T., (2018)]

Insights from CFT: contour for the EE





- A is a half-line [Cardy, E.T., (2016)]

$$s_{\rm h.l.}(x,t) = \frac{c}{6}\, \mathcal{F}_{\rm h.l.}(x,t)$$

$$\mathcal{F}_{\rm h.l.}(x,t) \equiv \frac{2\pi \,[\cosh(2\pi t/\tau_0)]^2 \coth(\pi x/\tau_0)}{\tau_0\,[\cosh(4\pi t/\tau_0) + \cosh(2\pi x/\tau_0)]}$$

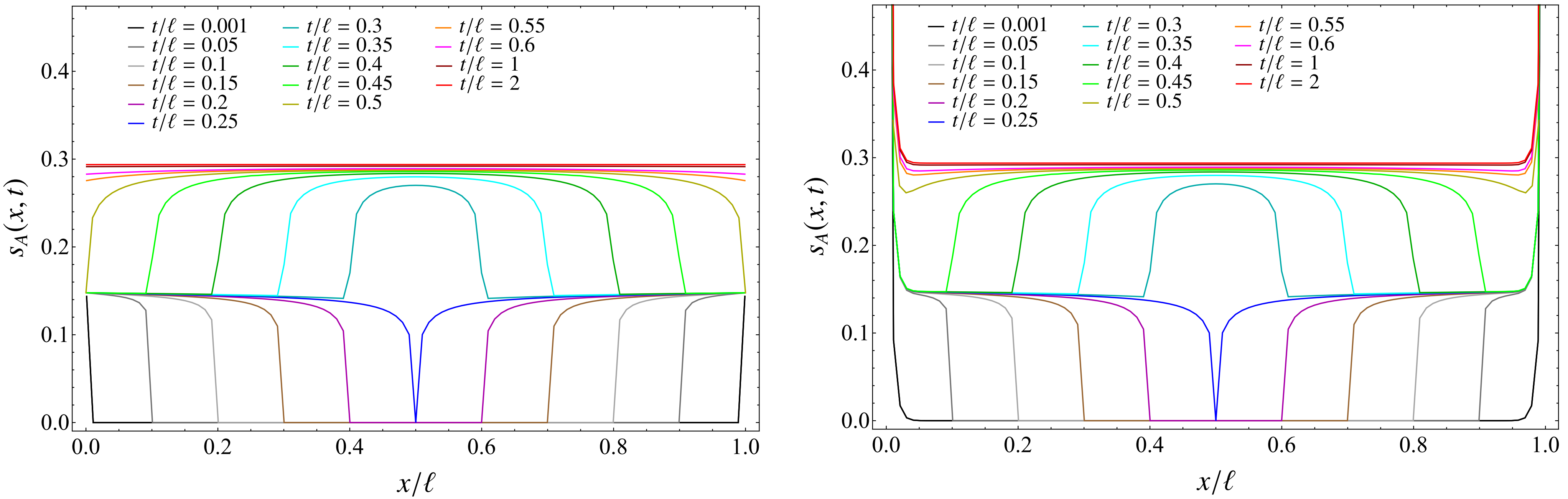
- Naive guess when A is an interval [Di Giulio, Arias, E.T., (2019)]

$$s_{A,0}(x,t) \equiv s_{\rm h.l.}(x,t) + s_{\rm h.l.}(\ell - x, t)$$

- Integrals of the contour function $(x_1, x_2) \subseteq A$

$$\mathcal{S}_{A,0}(x_1,x_2;t) \equiv \int_{x_1}^{x_2} s_{A,0}(x,t)\, dx$$

Contour for EE in the HC from quasi-particle picture



- By exploiting the quasi-particle picture of [Calabrese, Cardy, (2005)] we find [Di Giulio, Arias, E.T., (2019)]

$$s_A(x,t) = \frac{1}{2}\left[\int_{x<2|v_p|t<\ell} \tilde{s}(p)\, dp + \int_{\ell - x<2|v_p|t<\ell} \tilde{s}(p)\, dp\right] + \int_{2|v_p|t>\ell} \tilde{s}(p)\, dp + f_0(x)$$

For the harmonic chain we used $\tilde{s}(p)$ obtained in [Alba, Calabrese, (2018)]

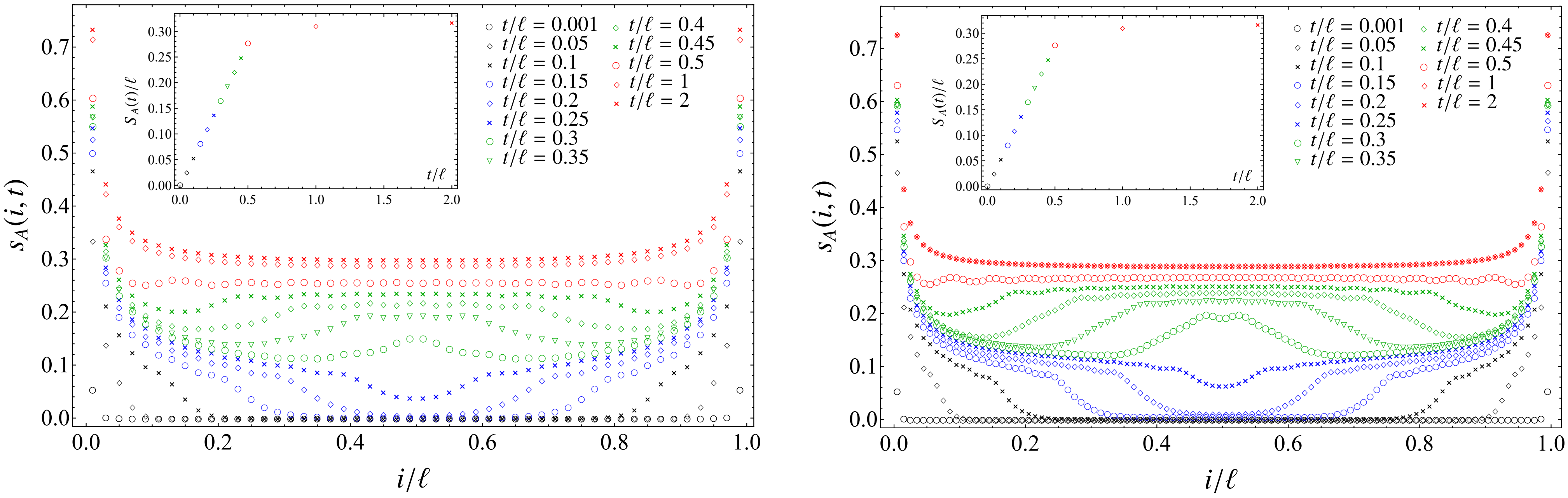
- Also $\mathcal{S}_A(x_1, x_2)$ can be written in a similar form

Contour for EE in the HC after a global quench

- Contour function after quantum quenches in fermionic chains
  [Chen, Vidal, (2014)]

- Contour function in the harmonic chain after a quench $\omega_0 = 1 \to \omega = 0$
  [Di Giulio, Arias, E.T., (2019)]

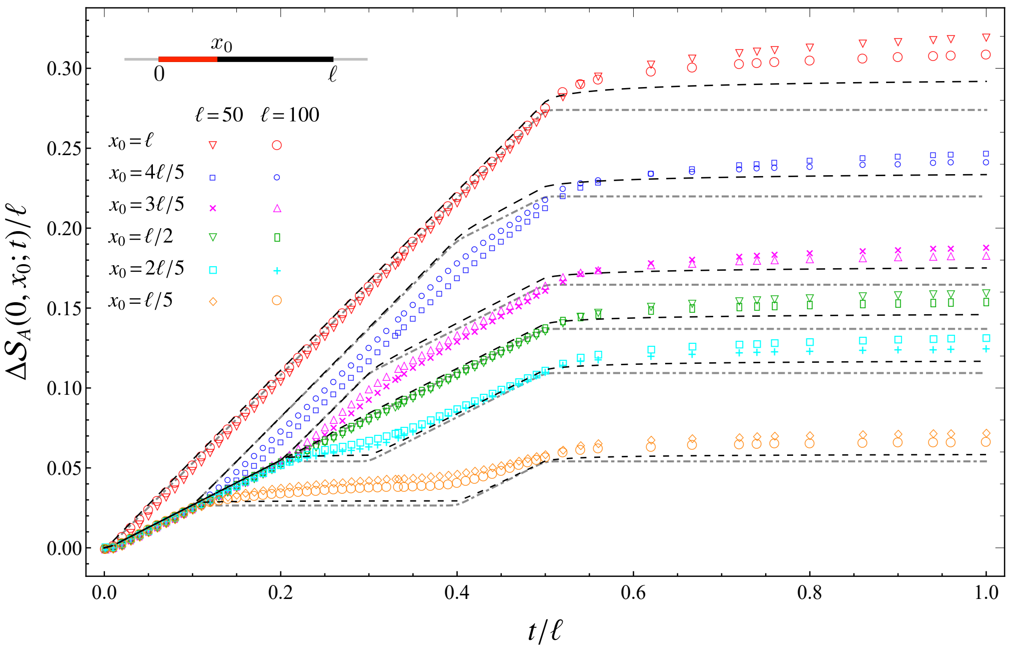
$\ell = 50$ $\qquad\qquad$ $\ell = 100$

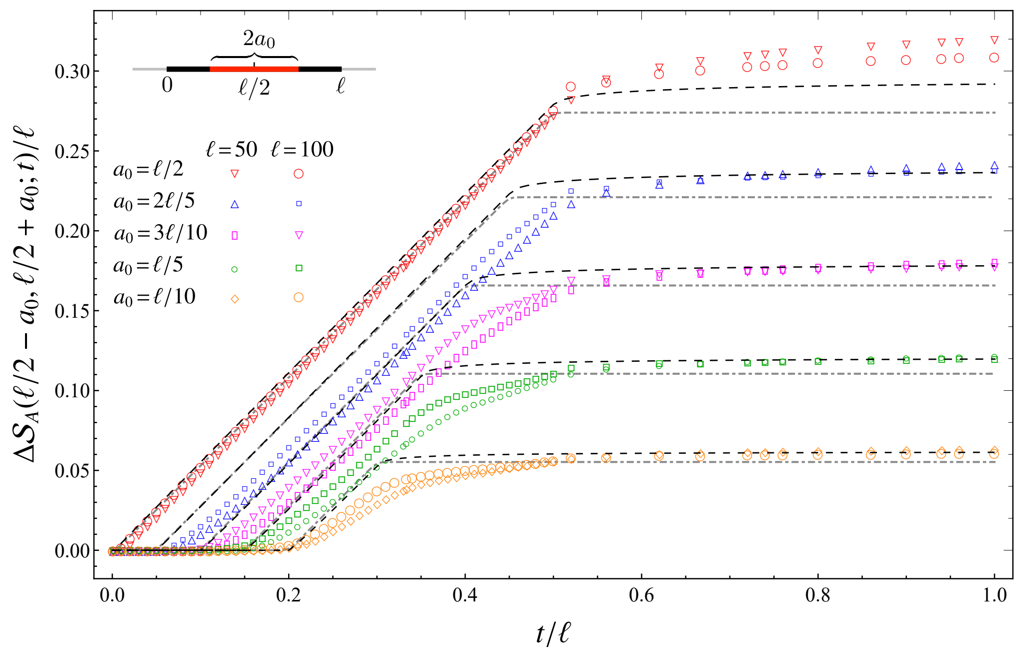


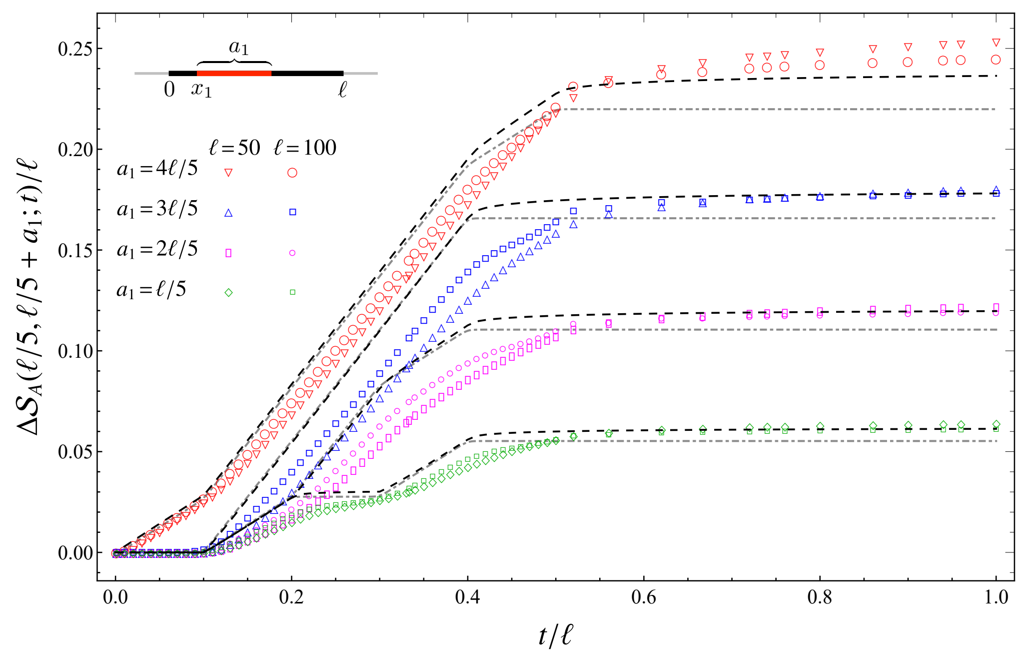
- Same qualitative behaviour observed for fermions [Chen, Vidal, (2014)]

Integrals of the contour for EE in the HC

[Di Giulio, Arias, E.T., (2019)]







EH matrix & contour for the EE in the FFC

- The EH matrix T can be written in terms of the correlation matrix C_A restricted to the interval [Peschel, (2003)]

$$\widehat{K}_A = \sum_{i,j=1}^{\ell} T_{i,j}\, \hat{c}_i^\dagger\, \hat{c}_j$$

$$T^{\mathrm{t}} = \log\big(C_A(t)^{-1} - \mathbf{1}\big)$$

$$T_{i,j} = \sum_{k=1}^{\ell} \eta_k\, \widetilde{U}^*_{k,i}\, \widetilde{U}_{k,j}$$

We consider the global quench of [Eisler, Peschel, (2007)]

- The matrix T is complex after the quench

- The contour function $s_A(i)$ depends also on $\widetilde{U}_{k,i}$ [Chen, Vidal, (2014)]

$$S_A = \sum_{k=1}^{\ell} s(\zeta_k) \qquad \eta_k = \log(1/\zeta_k - 1)$$

$$S_A = \sum_{i\,\in\,A} s_A(i)$$

$$s_A(i) = \sum_{k=1}^{\ell} p_k(i)\, s(\zeta_k)$$

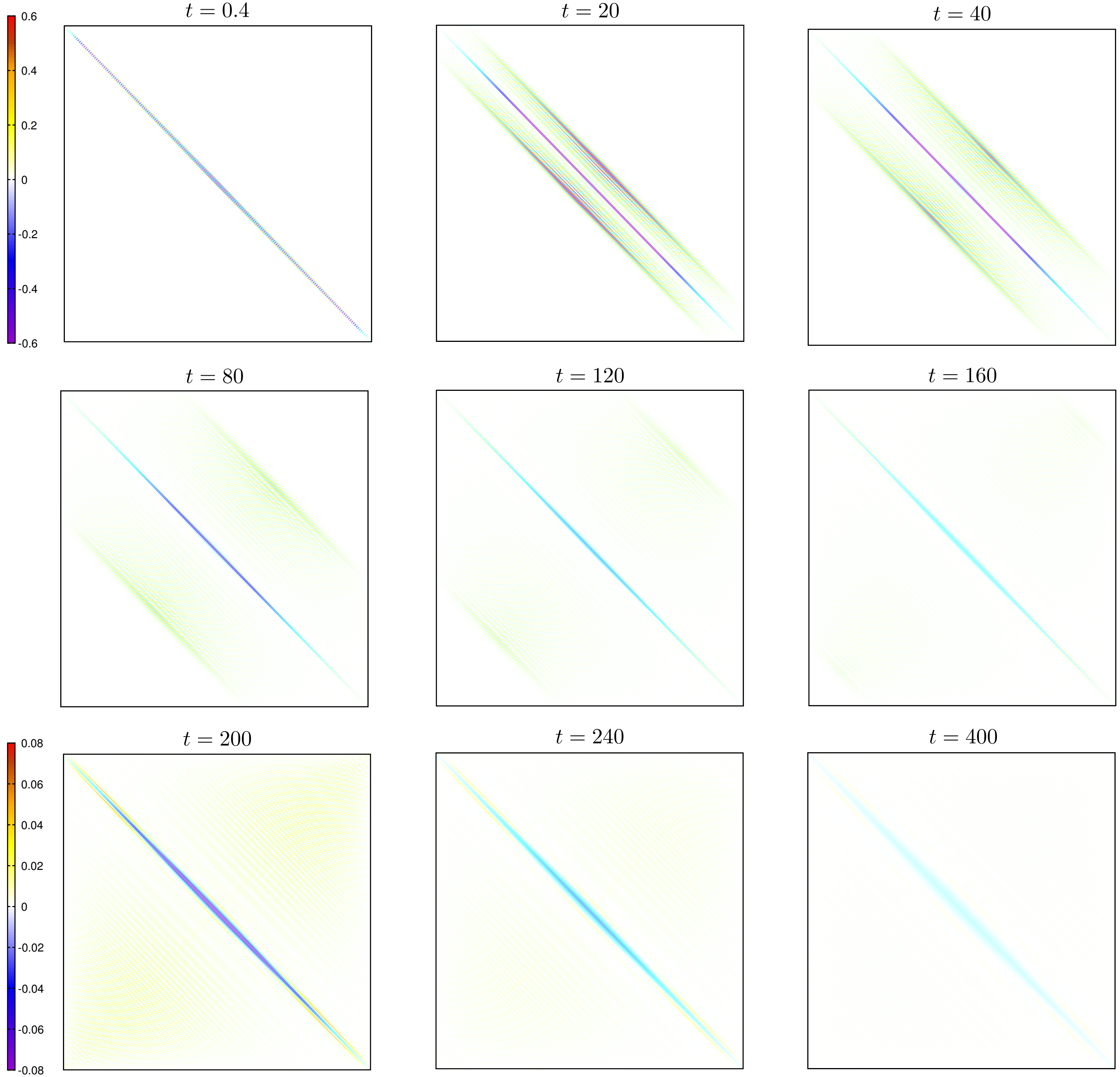
$$\begin{cases} p_k(i) = \big|\widetilde{U}_{k,i}\big|^2 \\ \displaystyle\sum_{i=1}^{\ell} p_k(i) = 1 \end{cases}$$

$$\mathcal{S}_A(i_1, i_2) = \sum_{i=i_1}^{i_2} s_A^{(n)}(i) \qquad i_1, i_2 \in A$$

Evolution of the EH matrix (real part) in the FFC

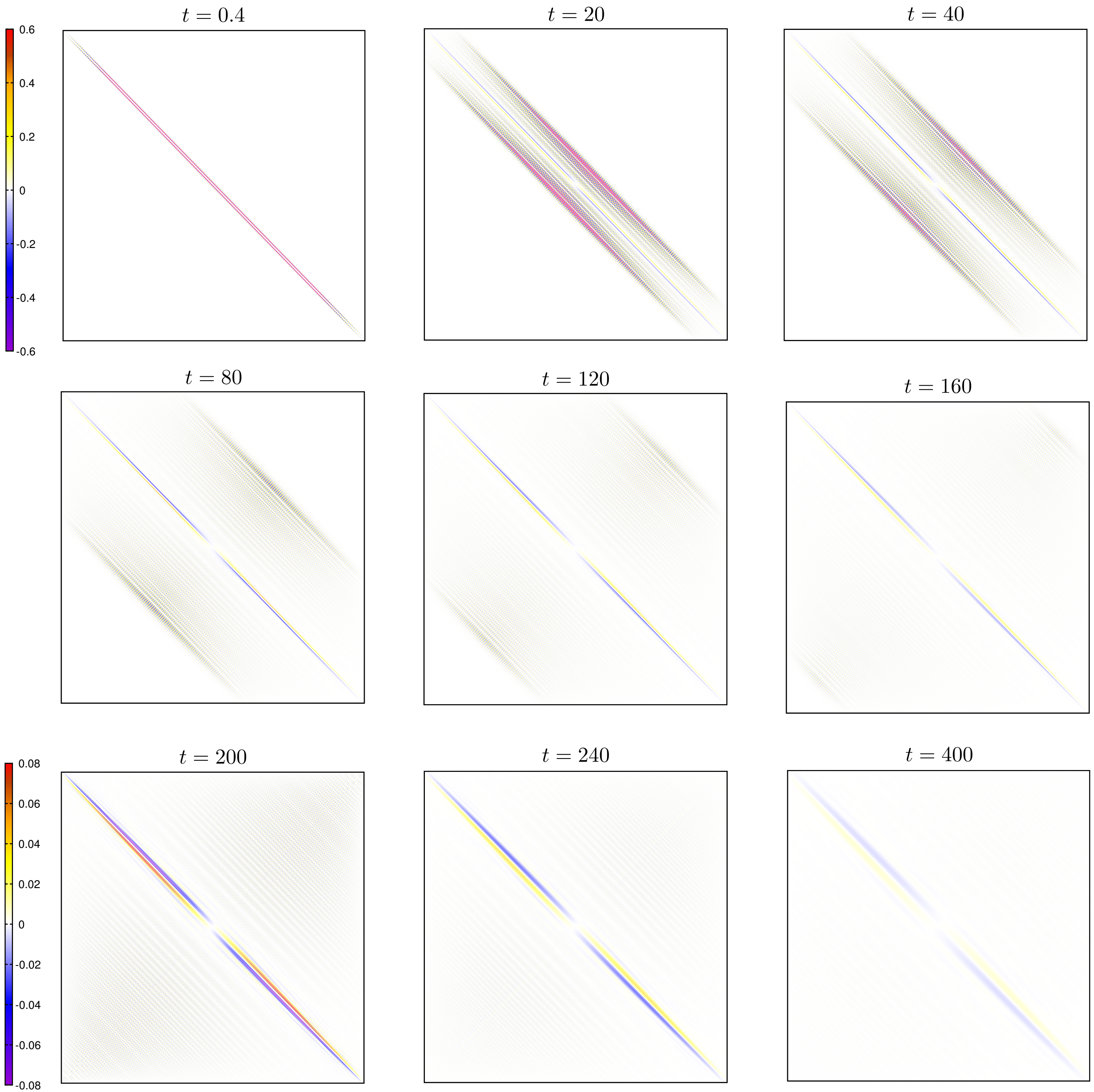
[Di Giulio, Arias, E.T., (2019)]

$\ell = 400$



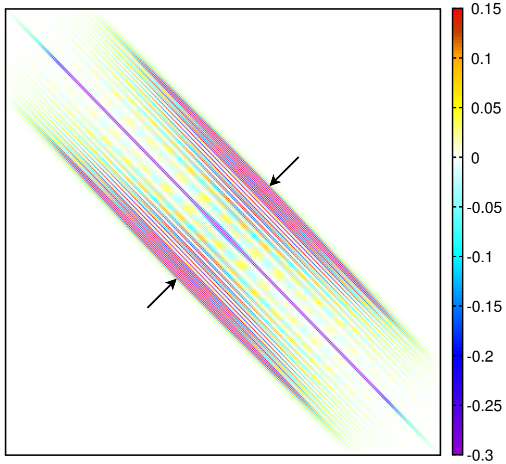
Evolution of the EH matrix (imaginary part) in the FFC

$\ell = 400$

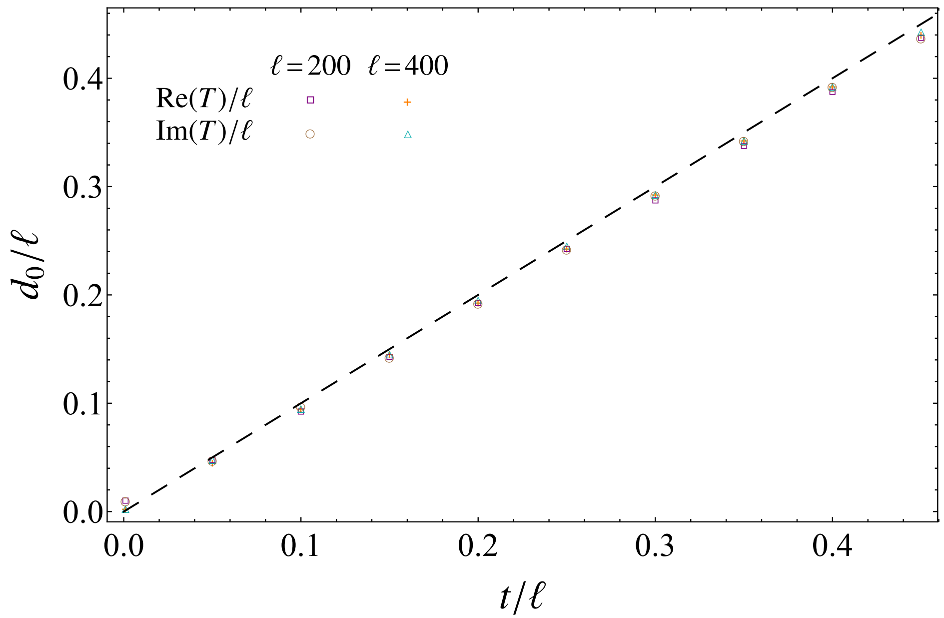


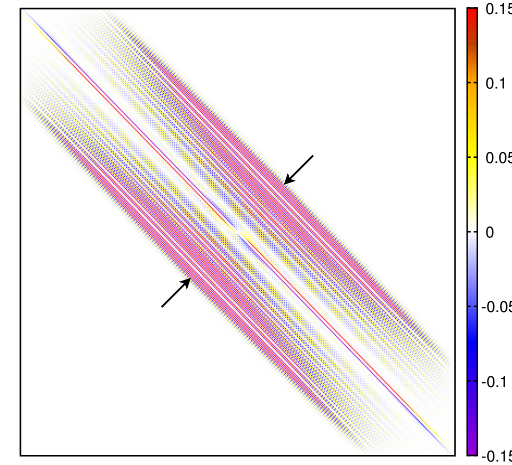
EH matrix in FFC: antidiagonals

- Linear growth of the antidiagonals of Re(T) and Im(T)



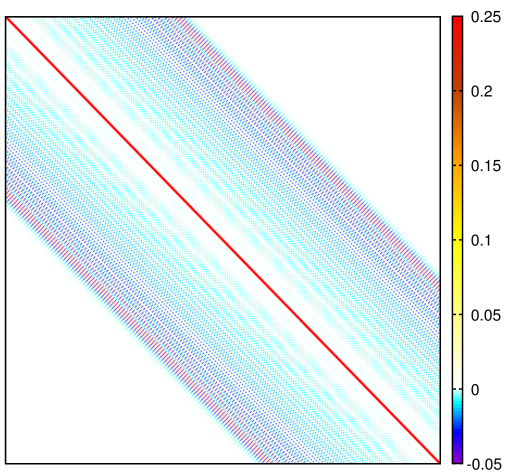
Re(T)





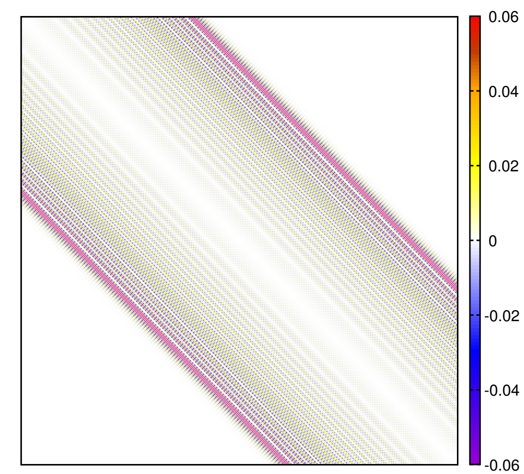
Im(T)

- Same linear growth observed for antidiagonals of $C_A(t)$

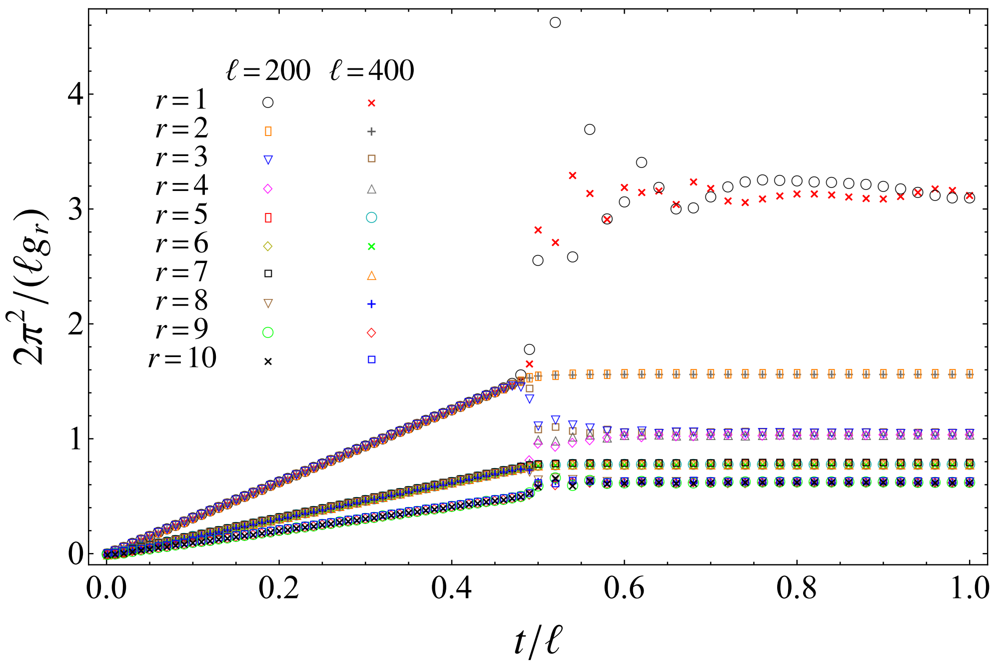


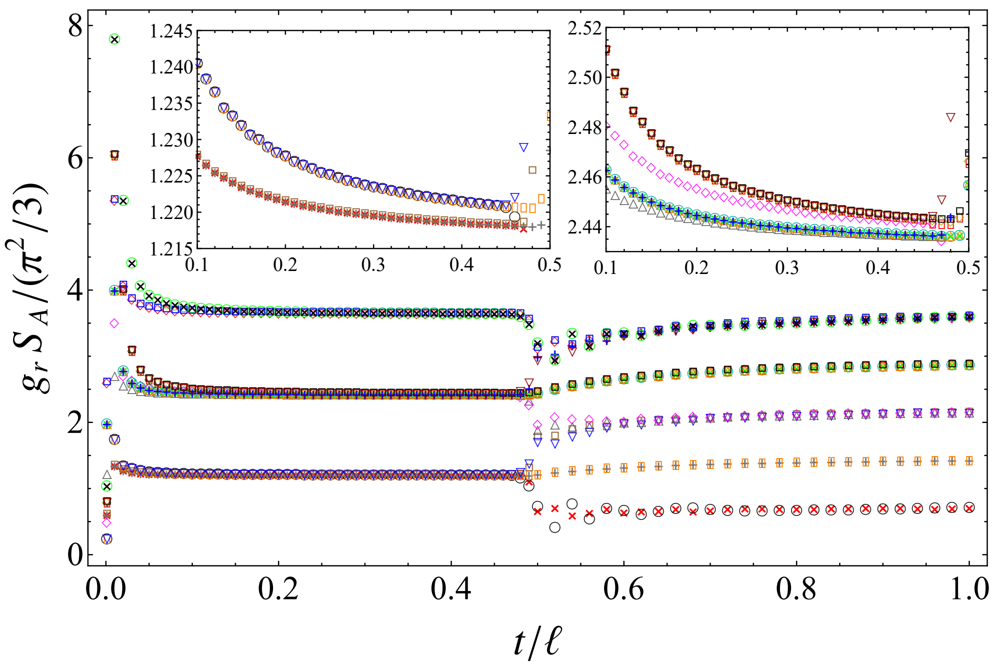
Re(C_A)

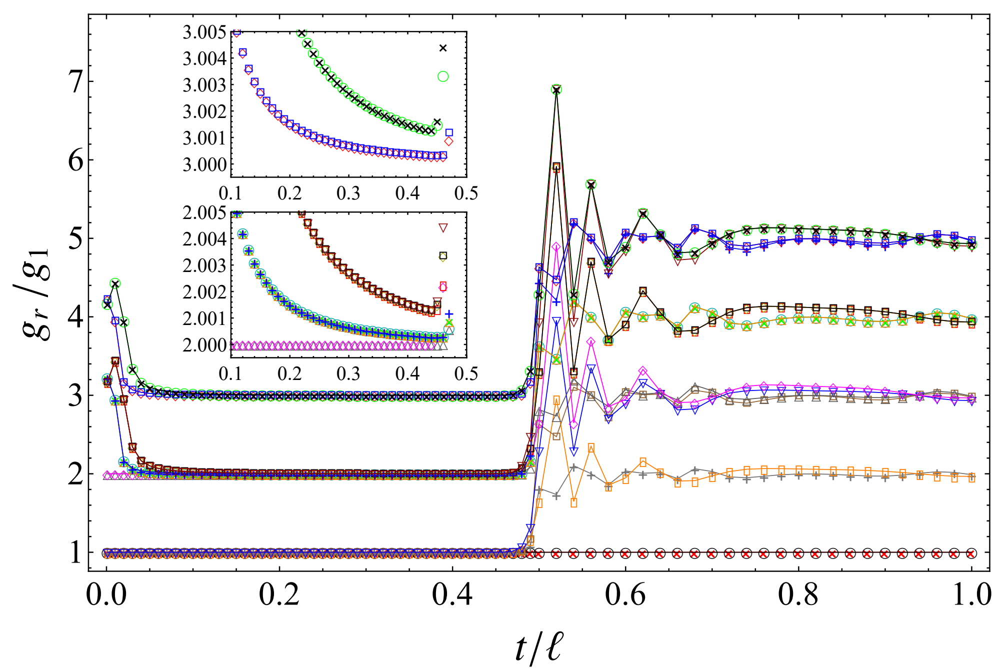
Im(C_A)



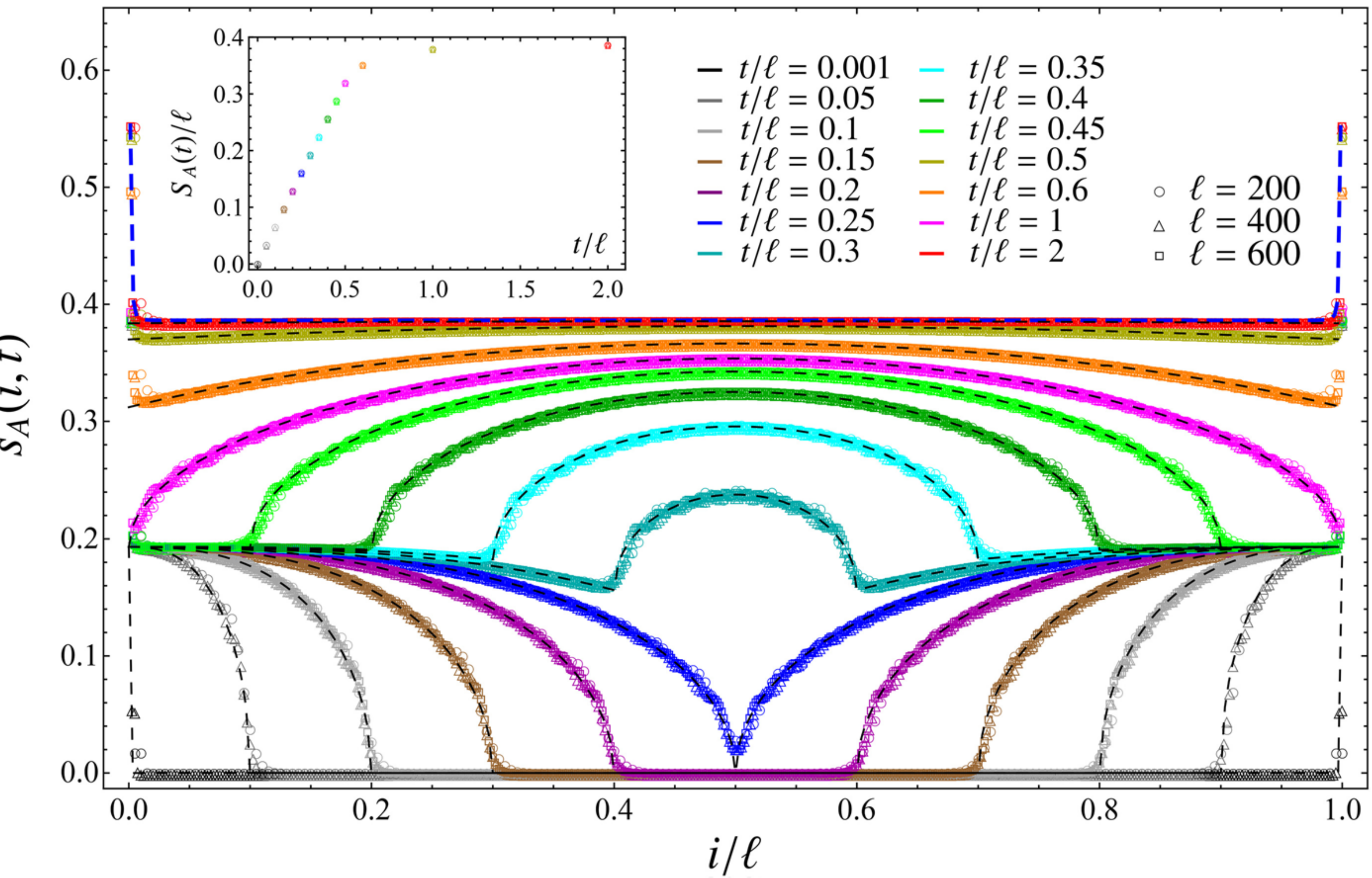
Gaps in the entanglement spectrum in the FFC



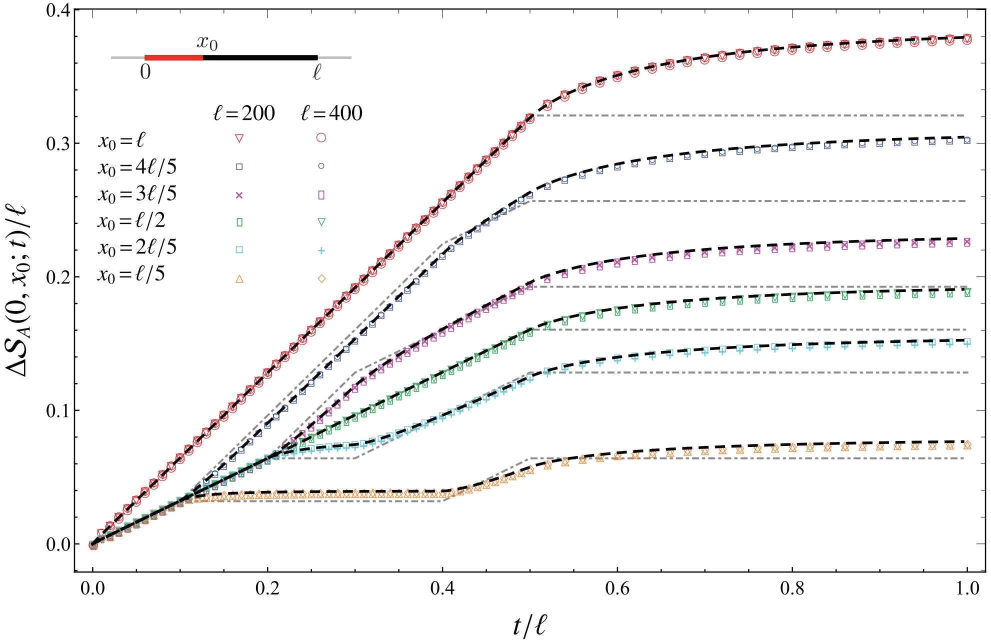


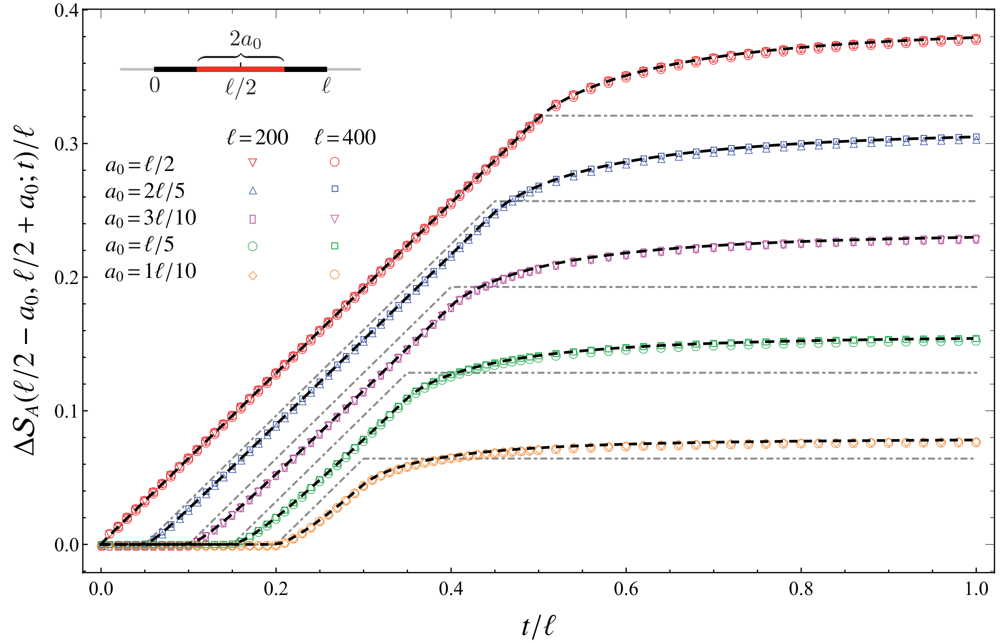


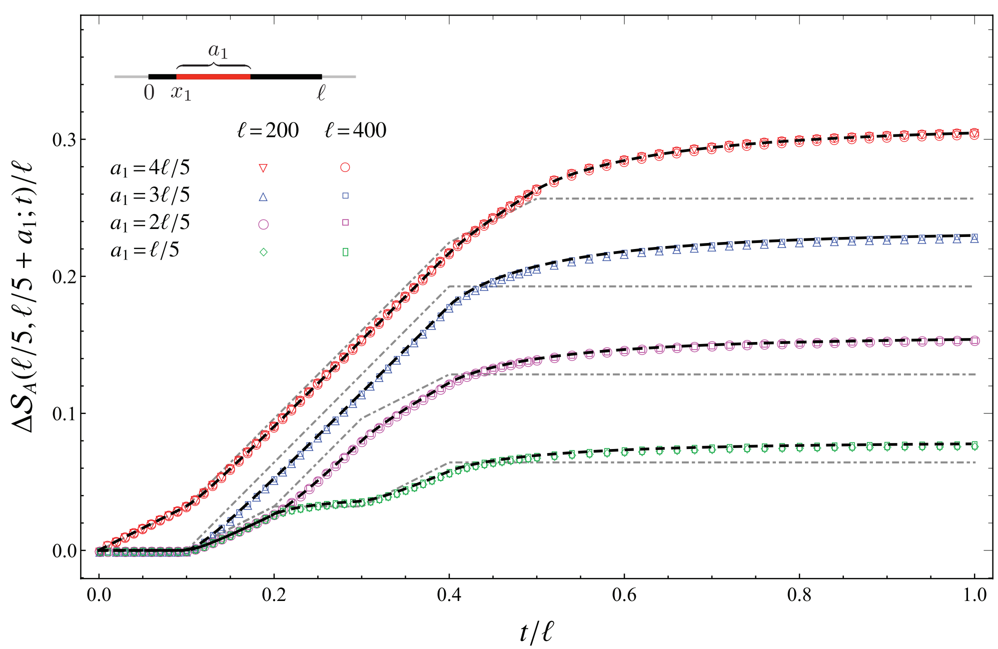
Contour for the EE in the FFC



Integrals of the contour for EE in the HC







Conclusions & some open issues

- A BCFT approach allows to study Entanglement Hamiltonians and entanglement spectra
- Continuum limit of the EH of an interval in free chains
- Entanglement Hamiltonians of an interval & contour for EE after a global quench:
  - Harmonic chain: quench of the frequency parameter
  - Chain of free fermions: a quench of the couplings
  - Analytic insights from CFT and quasi-particle picture

- Some open problems:
  - Other models [Roy, Pollmann, Saleur, (2020)]
  - Other quenches
  - Higher dimensions
  - Other spatial configurations
  - Holography

Thank you!