

Central Charges of Two-Dimensional Boundaries and Defects

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SOCIETY



Science & Technology
Facilities Council

Exploration of Duality, Geometry, and Entanglement
Le Studium Virtual Meeting
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Based on 1509.02160, 1812.00923, 1812.08745, 2003.02857



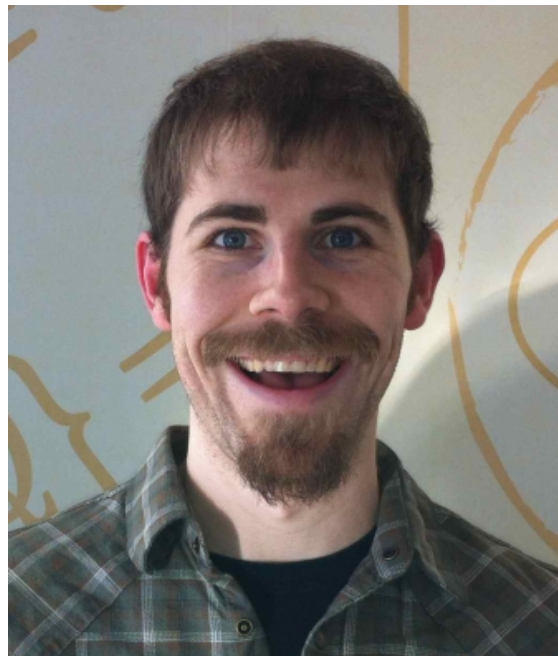
Adam Chalabi
Southampton



John Estes
SUNY Old Westbury



Darya Krym
NYC College of Technology



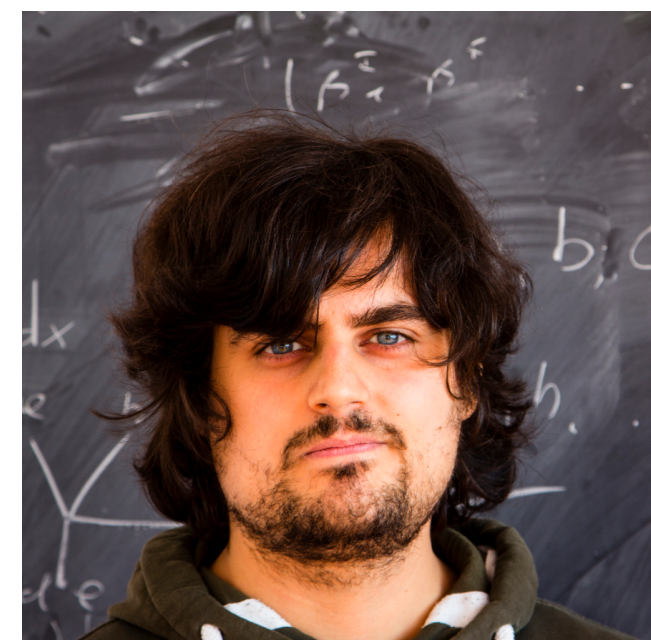
Kristan Jensen
San Francisco State



Brandon Robinson
Leuven



Ronnie Rodgers
Utrecht

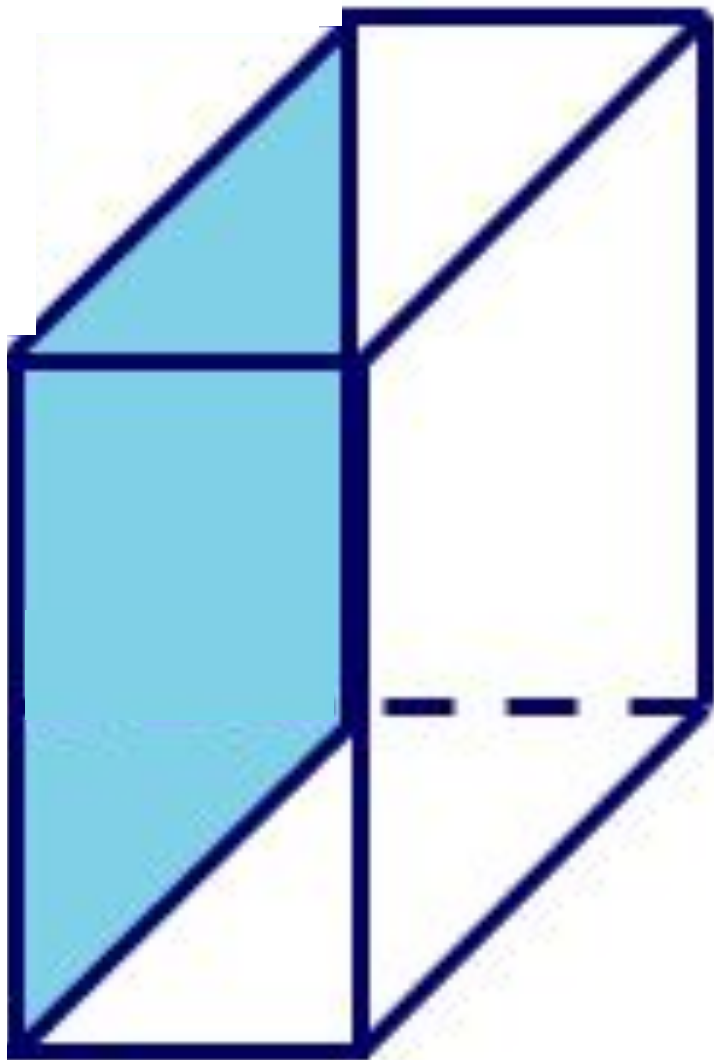


Jacopo Sisti
Southampton

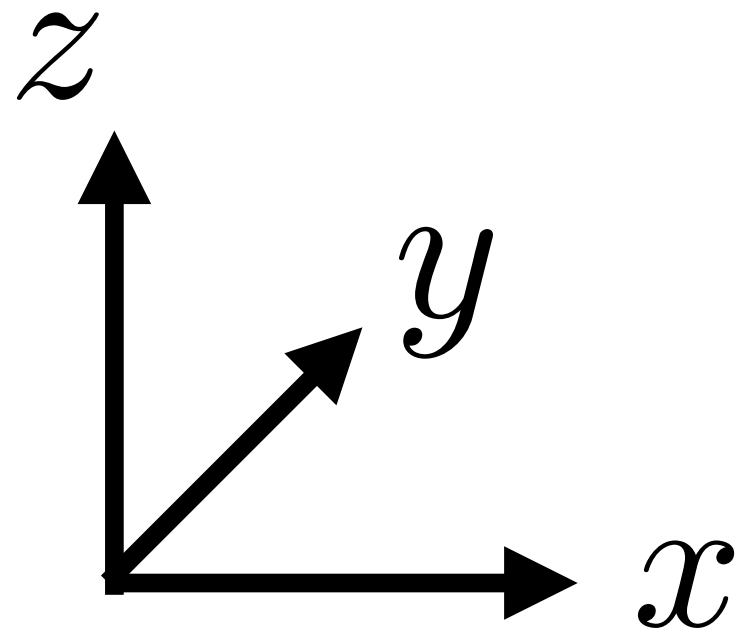
Central Charges of Two-Dimensional Boundaries and Defects

conformal field theory (CFT) with a boundary
+ conformally-invariant boundary conditions
possibly + massless degrees of freedom at the boundary

Boundary Conformal Field Theory (BCFT)



$$d = 3 \text{ BCFT}$$



Examples

critical Ising model in $d = 3$ with a boundary

graphene with a boundary

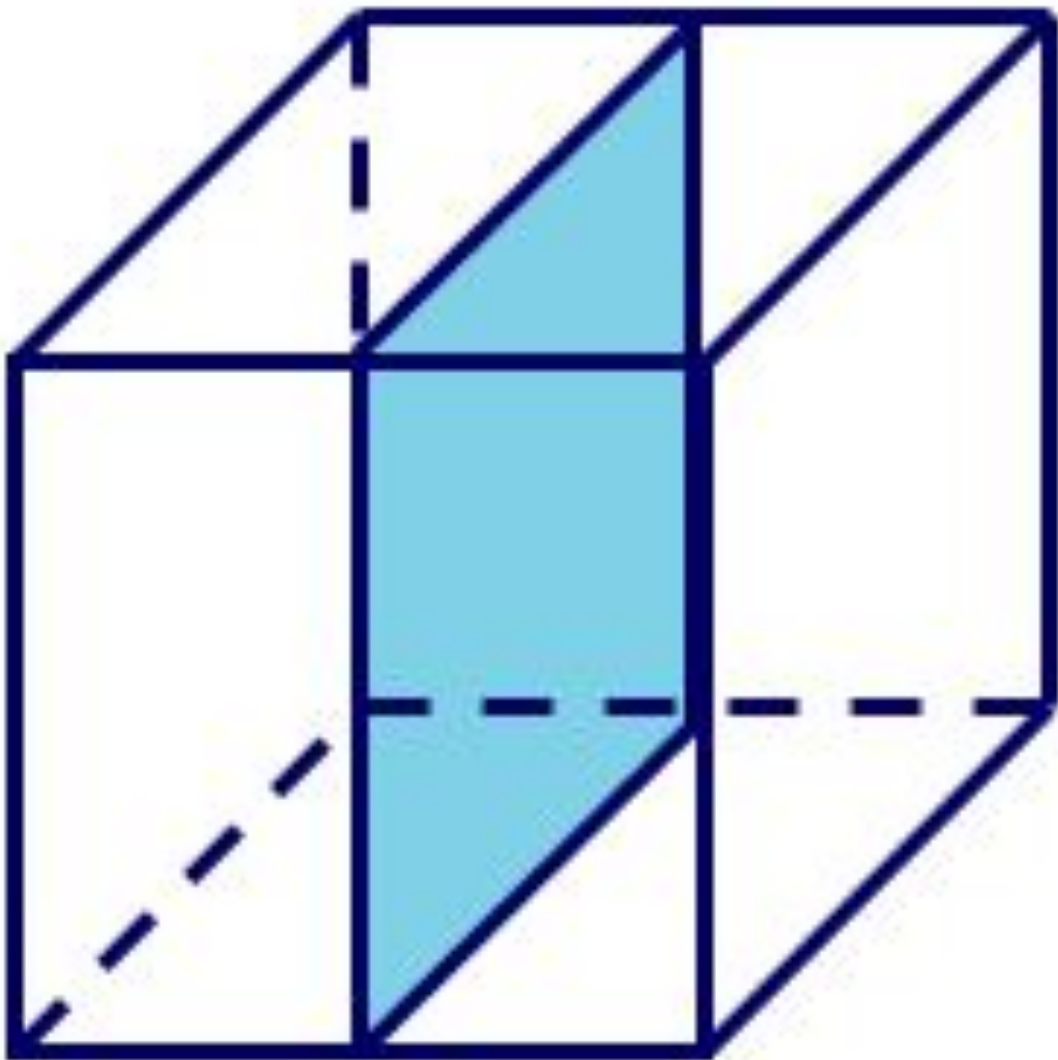
M-theory: M2s ending on M5s

string theory: various branes ending on branes

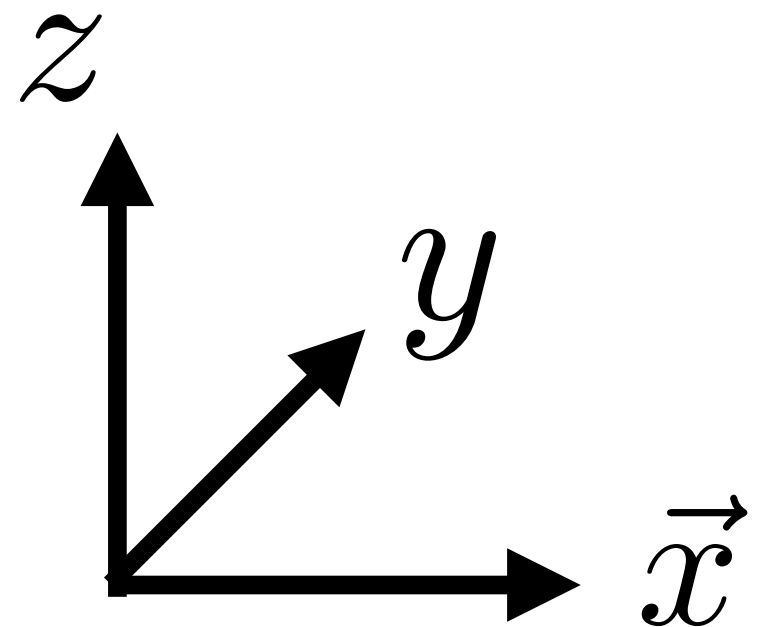
holographic examples

CFT with a conformally-invariant defect:
conformally-invariant boundary conditions along a submanifold
and/or massless degrees of freedom supported along a submanifold

Defect Conformal Field Theory (DCFT)



$$d \geq 3 \text{ DCFT}$$



Examples

critical Ising model in $d = 3$ with a domain wall

graphene with a line defect

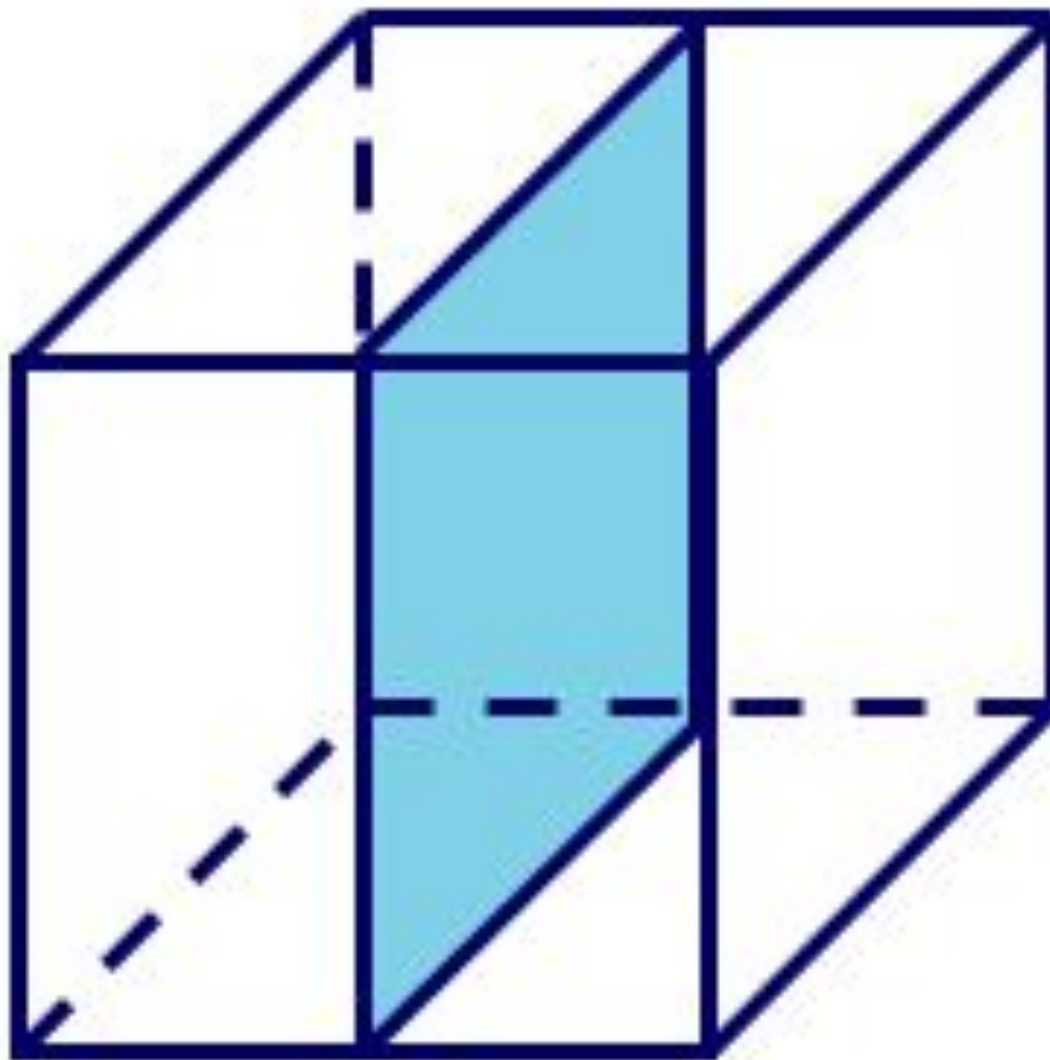
M-theory: M2s ending on M5s

string theory: various branes intersecting branes

holographic examples

Question

Can we define a central charge
for $d = 2$ boundaries and defects?



Central Charge

$$d = 2 \text{ CFT}$$

Virasoro algebra $\Rightarrow$ central charge $\mathcal{C}$

Counts the number of degrees of freedom

The c-theorem

A.B. Zamolodchikov
JETP Vol. 43 No. 12 p. 565, 1986

RG flow from UV CFT to IR CFT

$$\mathcal{C}_{UV} \geq \mathcal{C}_{IR}$$

Euclidean symmetry (Poincaré symmetry)

Locality

Reflection positivity (Unitarity)

Central Charge

$$d = 2 \text{ CFT}$$

Virasoro algebra $\Rightarrow$ central charge $\mathcal{C}$

Stress tensor 2-pt. function

$$\langle T_{zz}(z) T_{zz}(0) \rangle = \frac{c/2}{z^4}$$

Thermodynamic entropy

$$S_{\text{thermo}} = \frac{\pi}{3} c L T + \dots$$

Trace Anomaly

$$T_{\mu}^{\mu} = \frac{c}{24\pi} R$$

Entanglement Entropy

$$S_{\text{EE}} = \frac{c}{3} \ln(\ell/\varepsilon) + \dots$$

Central Charge

$d = 2$ boundary or defect in $d \geq 3$ BCFT or DCFT

~~Virasoro algebra $\Rightarrow$ central charge $\mathcal{C}$~~

Stress tensor 2-pt. function

??????

Thermodynamic entropy

??????

Trace Anomaly

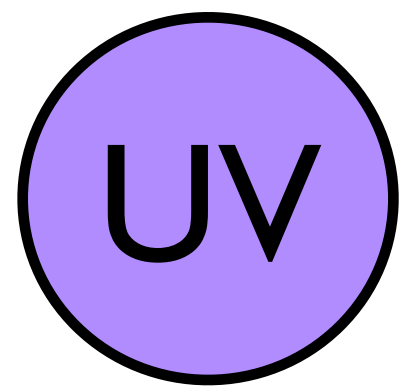
??????

Entanglement Entropy

??????

Outline:

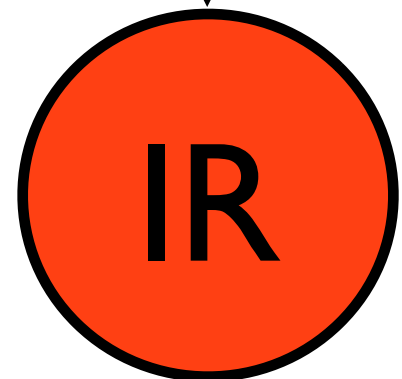
- 2d CFT Central Charge
- The Systems
- Boundary/Defect Central Charges
- Examples
- Summary and Outlook



Microscopic/High-energy scales

Quantum Field Theory

Renormalisation Group (RG) flow
from the UV to IR



Macroscopic/Low-energy scales

Quantum Field Theory

Generating functional Z

Non-dynamical background metric $g_{\mu\nu}(x)$

Stress-Energy Tensor

$$T_{\mu\nu} \equiv -\frac{2}{\sqrt{g}} \frac{\delta}{\delta g^{\mu\nu}} \ln Z[g_{\mu\nu}]$$

$$T_{\mu\nu} = T_{\nu\mu}$$

Quantum Field Theory

Generating functional Z

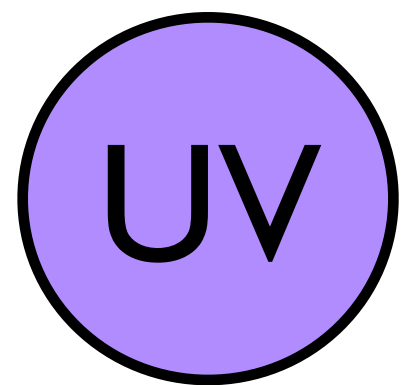
Non-dynamical background metric $g_{\mu\nu}(x)$

Stress-Energy Tensor

$$g_{\mu\nu} \rightarrow \delta_{\mu\nu}$$

Translational and Rotational Symmetry

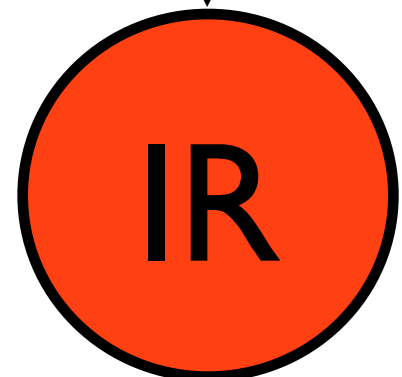
$$\partial_\mu T_{\mu\nu} = 0$$



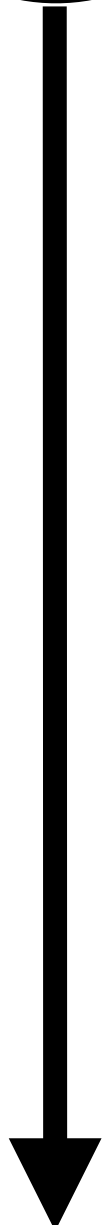
UV CFT

Conformal Field Theories
(CFTs)

RG fixed points



IR CFT



Conformal Field Theory

Generating functional Z

Non-dynamical background metric $g_{\mu\nu}(x)$

Conformal Transformation

Diffeomorphism

$$x^\mu \rightarrow x'^\mu(x)$$

such that

$$g_{\mu\nu}(x) \rightarrow e^{2\Omega(x)} g_{\mu\nu}(x)$$

Conformal Field Theory

$$g_{\mu\nu}(x) \rightarrow e^{2\Omega(x)} g_{\mu\nu}(x)$$

$$T_{\mu}^{\mu} = \frac{1}{\sqrt{g}} \frac{\delta}{\delta\Omega} \ln Z$$

Conformal invariance

$$T_{\mu}^{\mu} = 0$$

Conformal Field Theory

$$g_{\mu\nu}(x) \rightarrow \delta_{\mu\nu}$$

$$d > 2$$

Rotations

$$x^\mu \rightarrow \Lambda^\mu_\nu x^\nu$$

Translations

$$x^\mu \rightarrow x^\mu + c^\mu$$

Dilatations

$$x^\mu \rightarrow \lambda x^\mu$$

Special Conformal

$$x^\mu \rightarrow \frac{x^\mu + b^\mu x^2}{1 + 2x^\nu b_\nu + b^2 x^2}$$

$$SO(d+1, 1)$$

Conformal Field Theory

$$g_{\mu\nu}(x) \rightarrow \delta_{\mu\nu}$$

$$d = 2$$

complex coordinates $z, \bar{z}$

Conformal Transformation

$$z \rightarrow w(z) \qquad \bar{z} \rightarrow \bar{w}(\bar{z})$$

Conformal Field Theory

$$g_{\mu\nu}(x) \rightarrow \delta_{\mu\nu}$$

$$d = 2$$

Holomorphic and anti-holomorphic sectors DECOUPLE

$$[T_{\mu\nu}] = \begin{pmatrix} T_{zz} & T_{z\bar{z}} \\ T_{\bar{z}z} & T_{\bar{z}\bar{z}} \end{pmatrix}$$

$$T_{z\bar{z}} = T_{\bar{z}z}$$

$$T_{\mu}^{\mu} = T_{z\bar{z}} = T_{\bar{z}z}$$

Conformal Field Theory

$$g_{\mu\nu}(x) \rightarrow \delta_{\mu\nu}$$

$$d = 2$$

Holomorphic and anti-holomorphic sectors DECOUPLE

$$\partial_\mu T_{\mu\nu} = 0$$

$$\partial_z T_{\bar{z}\bar{z}} + \partial_{\bar{z}} T_{zz} = 0$$

$$T_\mu{}^\mu = T_{z\bar{z}} = T_{\bar{z}z} = 0$$

$$\partial_z T_{\bar{z}\bar{z}} = 0 \qquad \partial_{\bar{z}} T_{zz} = 0$$

Conformal Field Theory

$$T_{zz}(z) = \sum_{n=-\infty}^{\infty} \frac{L_n}{z^{n+2}} \quad T_{\bar{z}\bar{z}}(\bar{z}) = \sum_{n=-\infty}^{\infty} \frac{\bar{L}_n}{\bar{z}^{n+2}}$$

Virasoro algebra

$$[L_m, L_n] = (m - n)L_{n+m} + \frac{c}{12}m(m^2 - 1)\delta_{m+n,0}$$

$$[L_m, \bar{L}_n] = 0$$

$$m, n \in \mathbb{Z}$$

An infinite number of symmetry generators!

Conformal Field Theory

$$T_{zz}(z) = \sum_{n=-\infty}^{\infty} \frac{L_n}{z^{n+2}} \quad T_{\bar{z}\bar{z}}(\bar{z}) = \sum_{n=-\infty}^{\infty} \frac{\bar{L}_n}{\bar{z}^{n+2}}$$

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$$[L_m, \bar{L}_n] = 0$$

$$SO(d+1, 1) = SO(3, 1) \quad \text{subgroup} \quad \begin{array}{l} L_{\pm 1} \text{ and } L_0 \\ \bar{L}_{\pm 1} \text{ and } \bar{L}_0 \end{array}$$

Conformal Field Theory

$$T_{zz}(z) = \sum_{n=-\infty}^{\infty} \frac{L_n}{z^{n+2}} \quad T_{\bar{z}\bar{z}}(\bar{z}) = \sum_{n=-\infty}^{\infty} \frac{\bar{L}_n}{\bar{z}^{n+2}}$$

Virasoro algebra

$$[L_m, L_n] = (m - n)L_{n+m} + \frac{c}{12}n(m^2 - 1)\delta_{m+n,0}$$

$$[L_m, \bar{L}_n] = 0$$

$$SO(d+1, 1) = SO(3, 1)$$

subgroup

$$L_{\pm 1} \text{ and } L_0$$

$$\bar{L}_{\pm 1} \text{ and } \bar{L}_0$$

Central Charge

Counts the number of degrees of freedom

The c-theorem

RG flow from UV CFT to IR CFT

$$c_{UV} \geq c_{IR}$$

Reflection positivity (unitarity)
vacuum state normalizability

$$c \geq 0$$

Add a single, free, massless, real scalar field
or single, free, massless Dirac fermion

$$c \rightarrow c + 1$$

Central Charge

$$d = 2 \text{ CFT}$$

Virasoro algebra $\Rightarrow$ central charge $\mathcal{C}$

Stress tensor 2-pt. function

$$\langle T_{zz}(z) T_{zz}(0) \rangle = \frac{c/2}{z^4}$$

Thermodynamic entropy

$$S_{\text{thermo}} = \frac{\pi}{3} c L T + \dots$$

Trace Anomaly

$$T_{\mu}^{\mu} = \frac{c}{24\pi} R$$

Entanglement Entropy

$$S_{\text{EE}} = \frac{c}{3} \ln(\ell/\varepsilon) + \dots$$

Central Charge

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The c-theorem

A.B. Zamolodchikov
JETP Vol. 43 No. 12 p. 565, 1986

RG flow from UV CFT to IR CFT

$$c_{UV} \geq c_{IR}$$

Central Charge

Counts the number of degrees of freedom

Thermodynamic entropy

Cardy NPB 270 (186) 1986

System size L

Temperature T

$$T \gg 1/L$$

$$S_{\text{thermo}} = \frac{\pi}{3} c L T + \dots$$

Central Charge

Counts the number of degrees of freedom

Trace Anomaly

$$g_{\mu\nu} = \delta_{\mu\nu}$$

Conformal invariance

$$T_{\mu}^{\mu} = 0$$

Central Charge

Counts the number of degrees of freedom

Trace Anomaly

$$g_{\mu\nu} \neq \delta_{\mu\nu}$$

Quantum Effects
Break Conformal Invariance

$$T_{\mu}^{\mu} \neq 0$$

Central Charge

Counts the number of degrees of freedom

Trace Anomaly

CFT in any d

d odd

$$T_{\mu}^{\mu} = 0$$

d even

$$T_{\mu}^{\mu} \neq 0$$

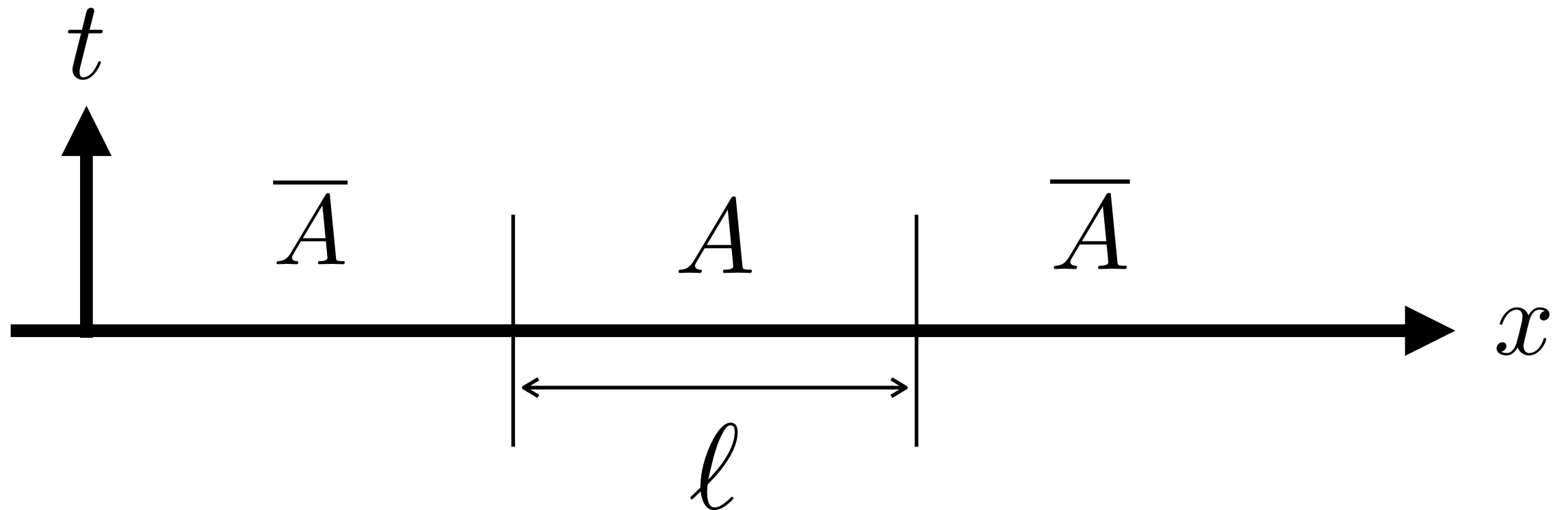
$d = 2$

$$T_{\mu}^{\mu} = \frac{c}{24\pi} R$$

Central Charge

Counts the number of degrees of freedom

Entanglement Entropy (EE)



$$\rho_A \equiv \text{tr}_{\overline{A}} \rho$$

$$S_{\text{EE}} = -\text{tr} \rho_A \ln \rho_A$$

Central Charge

Counts the number of degrees of freedom

Entanglement Entropy (EE)

$$d = 2 \text{ CFT}$$

Holzhey, Larsen, Wilczek hep-th/9403108 Calabrese + Cardy hep-th/0405152

interval of length ℓ

short-distance cutoff ε

$$S_{\text{EE}} = \frac{c}{3} \ln \frac{\ell}{\varepsilon} + \dots$$

Central Charge

$$d = 2 \text{ CFT}$$

Virasoro algebra $\Rightarrow$ central charge $\mathcal{C}$

Stress tensor 2-pt. function

$$\langle T_{zz}(z) T_{zz}(0) \rangle = \frac{c/2}{z^4}$$

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Entanglement Entropy

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Central Charge

$d = 2$ boundary or defect in $d \geq 3$ BCFT or DCFT

~~Virasoro algebra $\Rightarrow$ central charge $\mathcal{C}$~~

Stress tensor 2-pt. function

??????

Thermodynamic entropy

??????

Trace Anomaly

??????

Entanglement Entropy

??????

Outline:

- 2d CFT Central Charge
- The Systems
- Boundary/Defect Central Charges
- Examples
- Summary and Outlook

CFT

$d \geq 3$ flat space

Rotations

$$x^\mu \rightarrow \Lambda^\mu_\nu x^\nu$$

Translations

$$x^\mu \rightarrow x^\mu + c^\mu$$

Dilatations

$$x^\mu \rightarrow \lambda x^\mu$$

Special Conformal

$$x^\mu \rightarrow \frac{x^\mu + b^\mu x^2}{1 + 2x^\nu b_\nu + b^2 x^2}$$

$$SO(d+1, 1)$$

BCFT and DCFT

$d = 2$ boundary or defect in $d \geq 3$ flat space

Rotations

$$x^\mu \rightarrow \Lambda^\mu_\nu x^\nu$$

Translations

$$x^\mu \rightarrow x^\mu + c^\mu$$

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Special Conformal

$$x^\mu \rightarrow \frac{x^\mu + b^\mu x^2}{1 + 2x^\nu b_\nu + b^2 x^2}$$

$$SO(d+1, 1)$$

Broken to subgroup
that preserves the boundary or defect

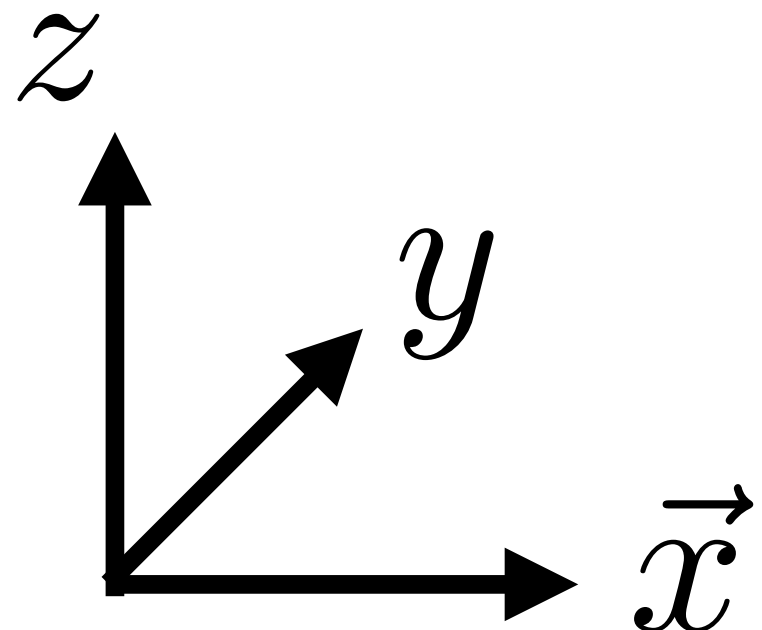
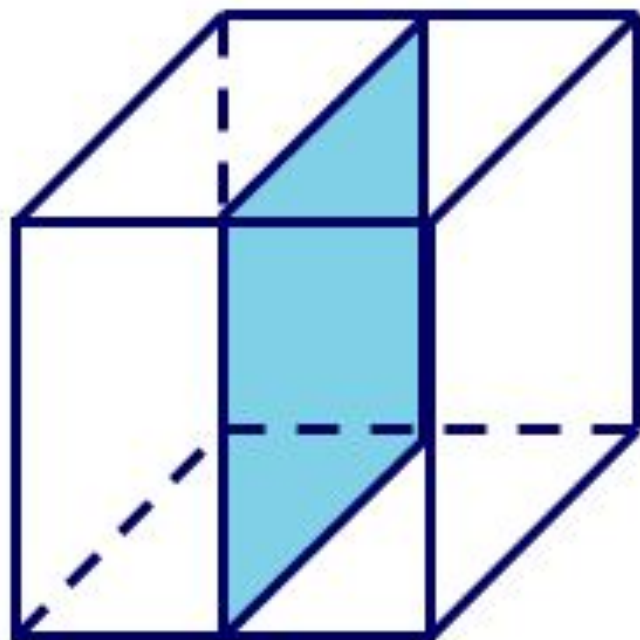
BCFT and DCFT

$d = 2$ boundary or defect in $d \geq 3$ flat space

Rotations

$$x^\mu \rightarrow \Lambda^\mu_\nu x^\nu$$

Broken to rotations in (y, z)
+ rotations in transverse directions



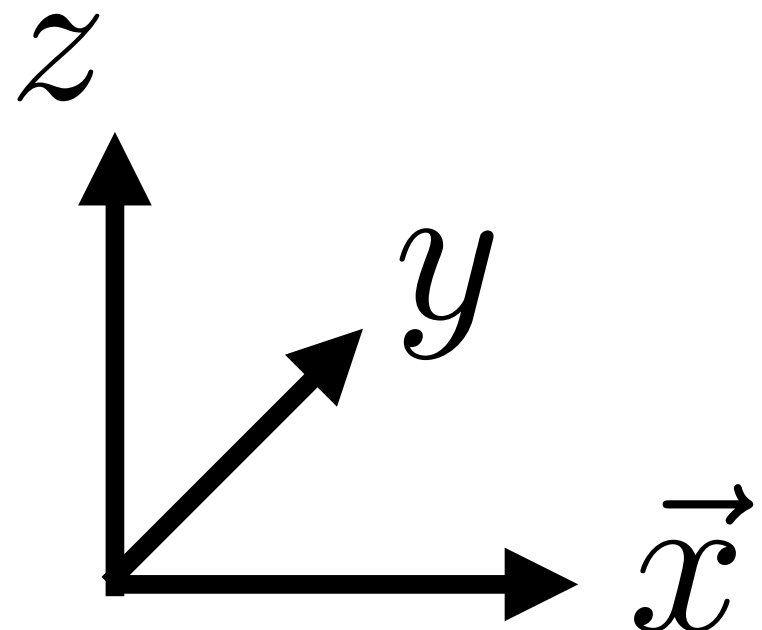
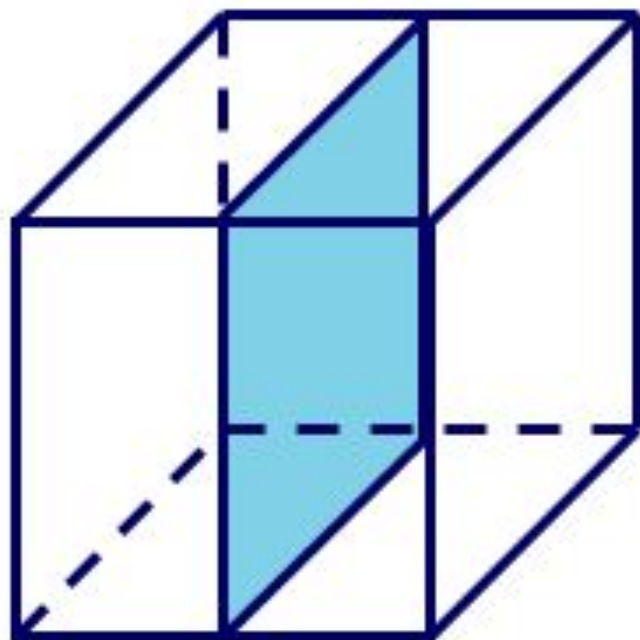
BCFT and DCFT

$d = 2$ boundary or defect in $d \geq 3$ flat space

Translations

$$x^\mu \rightarrow x^\mu + c^\mu$$

Broken to translations along (y, z)



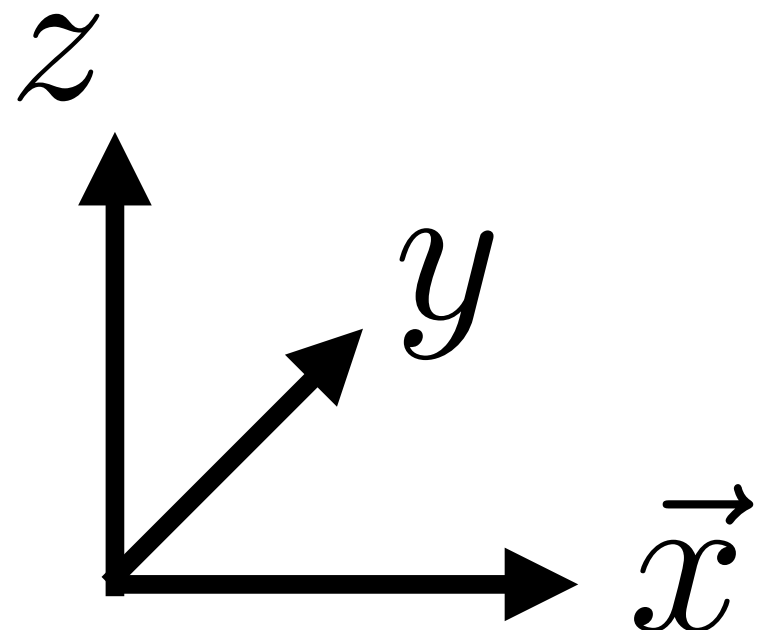
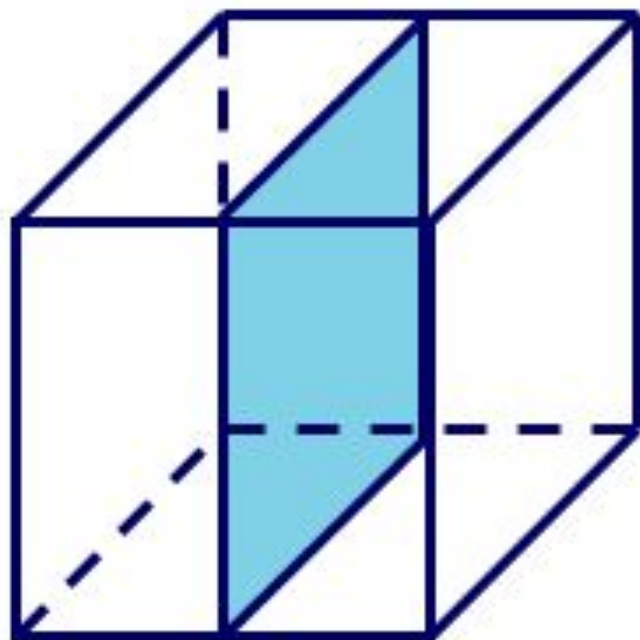
BCFT and DCFT

$d = 2$ boundary or defect in $d \geq 3$ flat space

Dilatations

$$x^\mu \rightarrow \lambda x^\mu$$

Unbroken



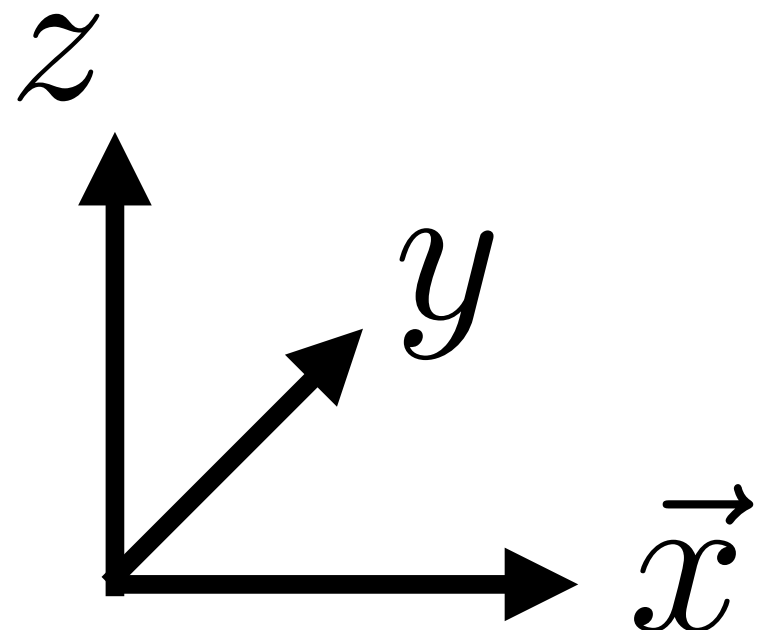
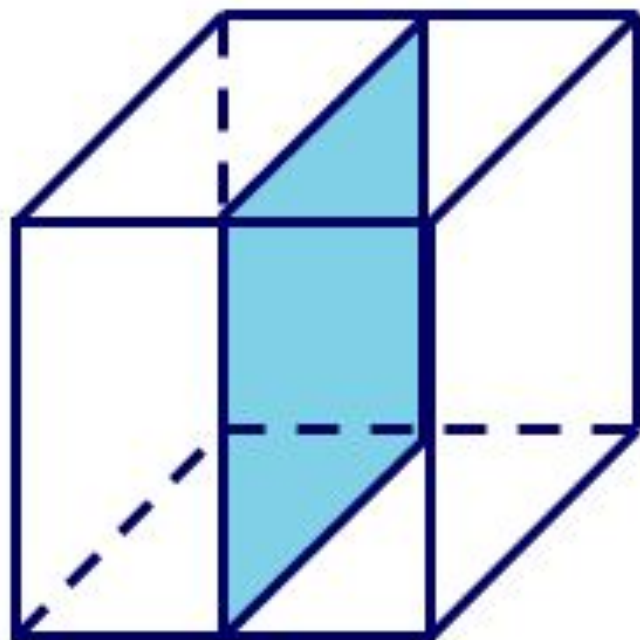
BCFT and DCFT

$d = 2$ boundary or defect in $d \geq 3$ flat space

Special Conformal

$$x^\mu \rightarrow \frac{x^\mu + b^\mu x^2}{1 + 2x^\nu b_\nu + b^2 x^2}$$

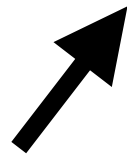
Broken to $b^\perp = 0$



BCFT and DCFT

$d = 2$ boundary or defect in $d \geq 3$ flat space

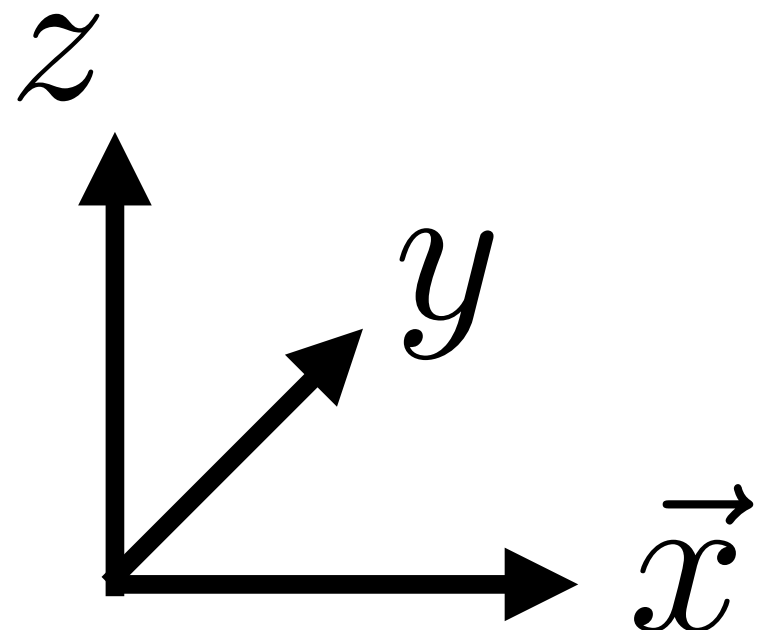
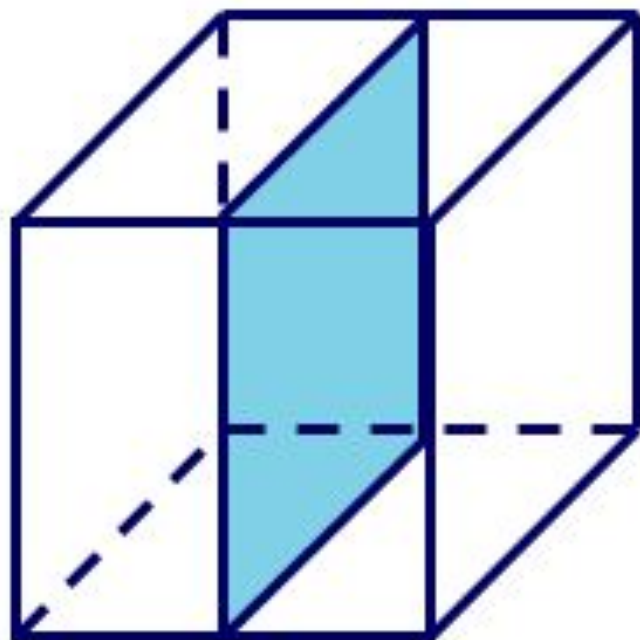
$$SO(d+1, 1) \rightarrow SO(3, 1) \times SO(d-2)$$



conformal transformations
preserving the boundary/defect



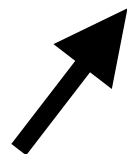
rotations around
the defect



BCFT and DCFT

$d = 2$ boundary or defect in $d \geq 3$ flat space

$$SO(d+1, 1) \rightarrow SO(3, 1) \times SO(d-2)$$



conformal transformations
preserving the boundary/defect

rotations around
the defect

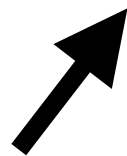
Generically NOT Virasoro!

A FINITE number of symmetry generators!

BCFT and DCFT

$d = 2$ boundary or defect in $d \geq 3$ flat space

$$SO(d+1, 1) \rightarrow SO(3, 1) \times SO(d-2)$$



conformal transformations
preserving the boundary/defect

rotations around
the defect

$$[L_m, L_n] = (m - n)L_{m+n} + \frac{c}{12}m(m^2 - 1)\delta_{m+n,0}$$

$$[\bar{L}_m, \bar{L}_n] = (m - n)\bar{L}_{m+n} + \frac{c}{12}m(m^2 - 1)\delta_{m+n,0}$$

$SO(3, 1)$ subgroup

$L_{\pm 1}$ and L_0
 $\bar{L}_{\pm 1}$ and $\bar{L}_0$

BCFT and DCFT

$d = 2$ boundary or defect in $d \geq 3$ flat space

Ward Identities

x^μ with $\mu = 1, 2, 3, \dots, d$

along the boundary/defect

x^a with $a = 1, 2$

transverse to the boundary/defect

x^i with $i = 3, 4, \dots, d$

BCFT and DCFT

$d = 2$ boundary or defect in $d \geq 3$ flat space

Ward Identities

$$\partial_\mu T^{\mu i} = \delta^{d-2} (x^j) D^i$$

Displacement operator D^i

$$T^{\mu\nu} = [T^{\mu\nu}]_{\text{bulk}} + \delta^{d-2} [T^{ab}]_{2d}$$

Gaussian pillbox integration around boundary/defect

$$D \propto [T^{\perp\perp}]_{\text{bulk}}^\partial$$

Boundary case:

$$\partial_a [T^{ab}]_{2d} \propto [T^{b\perp}]_{\text{bulk}}^\partial$$

BCFT and DCFT

$d = 2$ boundary or defect in $d \geq 3$ flat space

Ward Identities

$$\partial_\mu T^{\mu i} = \delta^{d-2} (x^j) D^i$$

Displacement operator D^i

$$T^{\mu\nu} = [T^{\mu\nu}]_{\text{bulk}} + \delta^{d-2} [T^{ab}]_{2d}$$

Conformal invariance

$$T_\mu{}^\mu = 0 \quad \Rightarrow \quad [T^\mu{}_\mu]_{\text{bulk}} = 0 \quad [T^a{}_a]_{2d} = 0$$

BCFT and DCFT

$d = 2$ boundary or defect in $d \geq 3$ flat space

Ward Identities

$$\partial_\mu T^{\mu i} = \delta^{d-2} (x^j) D^i$$

Displacement operator D^i

$$T^{\mu\nu} = [T^{\mu\nu}]_{\text{bulk}} + \delta^{d-2} [T^{ab}]_{2d}$$

Conformal invariance

$$\langle D^i(x^a) D^j(0) \rangle \propto \frac{\delta^{ij}}{|x^a|^4}$$

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Central Charge

$$d = 2 \text{ CFT}$$

Virasoro algebra $\Rightarrow$ central charge $\mathcal{C}$

Stress tensor 2-pt. function

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Thermodynamic entropy

$$S_{\text{thermo}} = \frac{\pi}{3} c L T + \dots$$

Trace Anomaly

$$T_{\mu}^{\mu} = \frac{c}{24\pi} R$$

Entanglement Entropy

$$S_{\text{EE}} = \frac{c}{3} \ln(\ell/\varepsilon) + \dots$$

Central Charge

$d = 2$ boundary or defect in $d \geq 3$ BCFT or DCFT

~~Virasoro algebra $\Rightarrow$ central charge $\mathcal{C}$~~

Stress tensor 2-pt. function

??????

Thermodynamic entropy

??????

Trace Anomaly

??????

Entanglement Entropy

??????

Central Charge

$d = 2$ boundary or defect in $d \geq 3$ BCFT or DCFT

~~Virasoro algebra $\Rightarrow$ central charge $\mathcal{C}$~~

Stress tensor 2-pt. function

??????

Billo, Goncalves, Lauria, Meineri
1601.02883

Prochazka
1804.01974

Thermodynamic entropy

??????

Central Charge

Counts the number of degrees of freedom

Trace Anomaly

$$g_{\mu\nu} = \delta_{\mu\nu}$$

Conformal invariance

$$T_{\mu}^{\mu} = 0$$

Central Charge

Counts the number of degrees of freedom

Trace Anomaly

$$g_{\mu\nu} \neq \delta_{\mu\nu}$$

Quantum Effects
Break Conformal Invariance

$$T_{\mu}^{\mu} \neq 0$$

Central Charge

Counts the number of degrees of freedom

Trace Anomaly

CFT in any d

d odd

$$T_{\mu}^{\mu} = 0$$

d even

$$T_{\mu}^{\mu} \neq 0$$

$d = 2$

$$T_{\mu}^{\mu} = \frac{c}{24\pi} R$$

Central Charge

$d = 2$ boundary or defect in $d \geq 3$ BCFT or DCFT

Trace Anomaly

$$T_{\mu}^{\mu} = [T_{\mu}^{\mu}]_{\text{bulk}} + \delta^{d-2} [T_{\mu}^{\mu}]_{2d}$$

What is $[T_{\mu}^{\mu}]_{2d}$?

Berenstein, Corrado, Fischler, Maldacena hep-th/9809188

Graham + Witten hep-th/9901021

Henningson + Skenderis hep-th/9905163

Gustavsson hep-th/0310037, 0404150

Asnin 0801.1469

Schwimmer + Theisen 0802.1017

Trace Anomaly

What is the general form of T_{μ}^{μ} ?

Step #1

Write down all curvature invariants
built from $g_{\mu\nu}$
with the correct dimension

$d = 4$ CFT

$$T_{\mu}^{\mu} = c_1 R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma} + c_2 R_{\mu\nu} R^{\mu\nu} + c_3 R^2 + c_4 \square R$$

Trace Anomaly

What is the general form of T_{μ}^{μ} ?

Step #2

Wess-Zumino consistency

$$g_{\mu\nu} \rightarrow e^{2\Omega_1} e^{2\Omega_2} g_{\mu\nu} = g_{\mu\nu} \rightarrow e^{2\Omega_2} e^{2\Omega_1} g_{\mu\nu}$$

Fixes some coefficients

$$T_{\mu}^{\mu} = c_1 R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma} + c_2 R_{\mu\nu} R^{\mu\nu} + c_3 R^2 + c_4 \square R$$

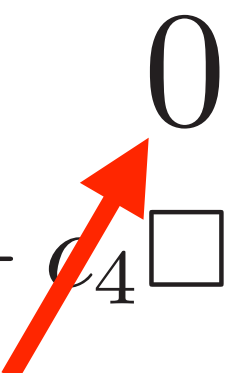
Trace Anomaly

What is the general form of T_{μ}^{μ} ?

Step #3

Add local counterterms
Determine how they enter T_{μ}^{μ}

Fixes more coefficients

$$T_{\mu}^{\mu} = c_1 R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma} + c_2 R_{\mu\nu} R^{\mu\nu} + c_3 R^2 + c_4 \square R$$


Trace Anomaly

CFT in any d

d odd

$$T_{\mu}^{\mu} = 0$$

d even

$$T_{\mu}^{\mu} \neq 0$$

$d = 2$

$$T_{\mu}^{\mu} = \frac{c}{24\pi} R$$

Trace Anomaly

$$d = 4 \quad \text{CFT}$$

$$T_{\mu}^{\mu} = a E - c W_{\mu\nu\rho\sigma} W^{\mu\nu\rho\sigma}$$

Euler density

$$E = R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma} - 4 R_{\mu\nu} R^{\mu\nu} + R^2$$

Weyl tensor

$$W_{\mu\nu\rho\sigma}$$

central charges a and c

Trace Anomaly

Wess-Zumino Consistency

$$g_{\mu\nu}(x) \rightarrow e^{2\Omega(x)} g_{\mu\nu}(x)$$

$$\int d^d x \sqrt{g} T_{\mu}^{\mu} \text{ is invariant}$$

$$T_{\mu}^{\mu} = a E - c W_{\mu\nu\rho\sigma} W^{\mu\nu\rho\sigma}$$

Type A

$$\sqrt{g} E$$

Type B

$$\sqrt{g} W^2$$

Changes by a total derivative

Invariant

Trace Anomaly

Wess-Zumino Consistency

Type A central charges are independent of marginal couplings

Osborn NPB 363 (1991) 486

$$d = 2$$

$$T_{\mu}^{\mu} = \frac{c}{24\pi} R$$

$$d = 4$$

$$T_{\mu}^{\mu} = a E - c W_{\mu\nu\rho\sigma} W^{\mu\nu\rho\sigma}$$

Central Charge

$d = 2$ boundary or defect in $d \geq 3$ BCFT or DCFT

Trace Anomaly

$$T_{\mu}^{\mu} = [T_{\mu}^{\mu}]_{\text{bulk}} + \delta^{d-2} [T_{\mu}^{\mu}]_{2d}$$

What is $[T_{\mu}^{\mu}]_{2d}$?

Berenstein, Corrado, Fischler, Maldacena hep-th/9809188

Graham + Witten hep-th/9901021

Henningson + Skenderis hep-th/9905163

Gustavsson hep-th/0310037, 0404150

Asnin 0801.1469

Schwimmer + Theisen 0802.1017

Geometry of Submanifolds

“worldsheet”

$$\sigma^1, \sigma^2$$

“target space”

$$x^\mu$$

embedding

$$x^\mu(\sigma^a)$$

induced metric

$$\hat{g}_{ab} = g_{\mu\nu} \frac{\partial x^\mu}{\partial \sigma^a} \frac{\partial x^\nu}{\partial \sigma^b}$$

Intrinsic curvature

$$\hat{R}_{abcd} \Rightarrow \hat{R}_{ab} \Rightarrow \hat{R}$$

Extrinsic curvature K_{ab}^μ

Mean curvature $K^\mu \equiv K_{ab}^\mu \hat{g}^{ab}$

Central Charge

$$[T_{\mu}^{\mu}]_{2d} = \frac{b}{24\pi} \hat{R} + \frac{d_1}{24\pi} (K_{ab}^{\mu} K_{\mu}^{ab} - \frac{1}{2} K^{\mu} K_{\mu}) - \frac{d_2}{24\pi} W^{ab}{}_{ab}$$

boundary/defect central charges

b

d_1

d_2

Central Charge

$$[T_\mu^\mu]_{2d} = \frac{b}{24\pi} \hat{R} + \frac{d_1}{24\pi} \left(K_{ab}^\mu K_\mu^{ab} - \frac{1}{2} K^\mu K_\mu \right) - \frac{d_2}{24\pi} W^{ab}_{ab}$$

Type A

Type B

$$\sqrt{\hat{g}} \hat{R}$$

$$\sqrt{\hat{g}} \left(K_{ab}^\mu K_\mu^{ab} - \frac{1}{2} K^\mu K_\mu \right)$$

$$\sqrt{\hat{g}} W_{ab}^{ab}$$

Changes by a
total derivative

Invariant

Central Charge

$$[T_{\mu}^{\mu}]_{2d} = \frac{b}{24\pi} \hat{R} + \frac{d_1}{24\pi} (K_{ab}^{\mu} K_{\mu}^{ab} - \frac{1}{2} K^{\mu} K_{\mu}) - \frac{d_2}{24\pi} W^{ab}_{ab}$$

Wess-Zumino Consistency

b is independent of boundary/defect marginal couplings

in general, b can depend on bulk marginal couplings

$d = 2$ $\mathcal{N} = (2, 0)$ supersymmetry

b is also independent of bulk marginal couplings

Central Charge

$$[T_{\mu}^{\mu}]_{2d} = \frac{b}{24\pi} \hat{R} + \frac{d_1}{24\pi} (K_{ab}^{\mu} K_{\mu}^{ab} - \frac{1}{2} K^{\mu} K_{\mu}) - \frac{d_2}{24\pi} W^{ab}_{ab}$$

b-theorem

Jensen and O'Bannon 1509.02160

Casini, Landea, Torroba 1812.08183

RG flow on the boundary/defect

$$b_{UV} \geq b_{IR}$$

Euclidean symmetry (Poincaré symmetry) along the boundary/defect

Locality

Reflection positivity (Unitarity)

Central Charge

$$[T_{\mu}^{\mu}]_{2d} = \frac{b}{24\pi} \hat{R} + \frac{d_1}{24\pi} (K_{ab}^{\mu} K_{\mu}^{ab} - \frac{1}{2} K^{\mu} K_{\mu}) - \frac{d_2}{24\pi} W^{ab}_{ab}$$

Bianchi, Meineri, Myers, Smolkin 1511.06713

Herzog + Huang 1707.06224

Herzog, Huang, Jensen 1709.07431

Displacement operator

$$\nabla_{\mu} T^{\mu i} = \delta^{d-2} (x^j) D^i$$

$$\langle D^i(x^a) D^j(0) \rangle \propto \frac{d_1 \delta^{ij}}{|x^a|^4}$$

Reflection positivity (unitarity)

$$d_1 \geq 0$$

Central Charge

$$[T_{\mu}^{\mu}]_{2d} = \frac{b}{24\pi} \hat{R} + \frac{d_1}{24\pi} (K_{ab}^{\mu} K_{\mu}^{ab} - \frac{1}{2} K^{\mu} K_{\mu}) - \frac{d_2}{24\pi} W^{ab}_{ab}$$

Conjecture

Herzog, Huang, Jensen | 709.0743 |

$$d = 3$$

$$b \geq -\frac{2}{3}d_1$$

Central Charge

$$[T_{\mu}^{\mu}]_{2d} = \frac{b}{24\pi} \hat{R} + \frac{d_1}{24\pi} (K_{ab}^{\mu} K_{\mu}^{ab} - \frac{1}{2} K^{\mu} K_{\mu}) - \frac{d_2}{24\pi} W^{ab}_{ab}$$

Weyl tensor

$$d = 3 \quad \Rightarrow \quad W_{\mu\nu\rho\sigma} \equiv 0$$

d_2 is only defined for $d > 3$

$\Rightarrow$ defect co-dimension > 1

Central Charge

$$[T_{\mu}^{\mu}]_{2d} = \frac{b}{24\pi} \hat{R} + \frac{d_1}{24\pi} (K_{ab}^{\mu} K_{\mu}^{ab} - \frac{1}{2} K^{\mu} K_{\mu}) - \frac{d_2}{24\pi} W^{ab}_{ab}$$

$SO(3, 1) \times SO(d - 2)$ symmetry implies

$$\langle T^{ab} \rangle = -\frac{h}{2\pi} \frac{\eta^{ab}}{|x^i|^d} \quad \langle T^{ai} \rangle = 0$$

$$\langle T^{ij} \rangle = \frac{h}{2\pi(d-3)} \frac{3\delta^{ij} |x^k|^2 - d x^i x^j}{|x^l|^{d+2}}$$

$$h \equiv \frac{1}{3 \text{vol}(S^{d-3})} \frac{d-3}{d-1} d_2$$

Bianchi, Meineri, Myers, Smolkin 1511.06713

Jensen, O'Bannon, Robinson, Rodgers 1812.08745

Central Charge

$$[T_{\mu}^{\mu}]_{2d} = \frac{b}{24\pi} \hat{R} + \frac{d_1}{24\pi} (K_{ab}^{\mu} K_{\mu}^{ab} - \frac{1}{2} K^{\mu} K_{\mu}) - \frac{d_2}{24\pi} W^{ab}_{ab}$$

Average Null Energy Condition (ANEC)

Faulkner, Leigh, Parrikar, Wang 1605.08072

Hartman, Kundu, Tajdini 1610.05308

along any null ray

$$\int_{-\infty}^{\infty} du \langle T_{uu} \rangle \geq 0$$

Central Charge

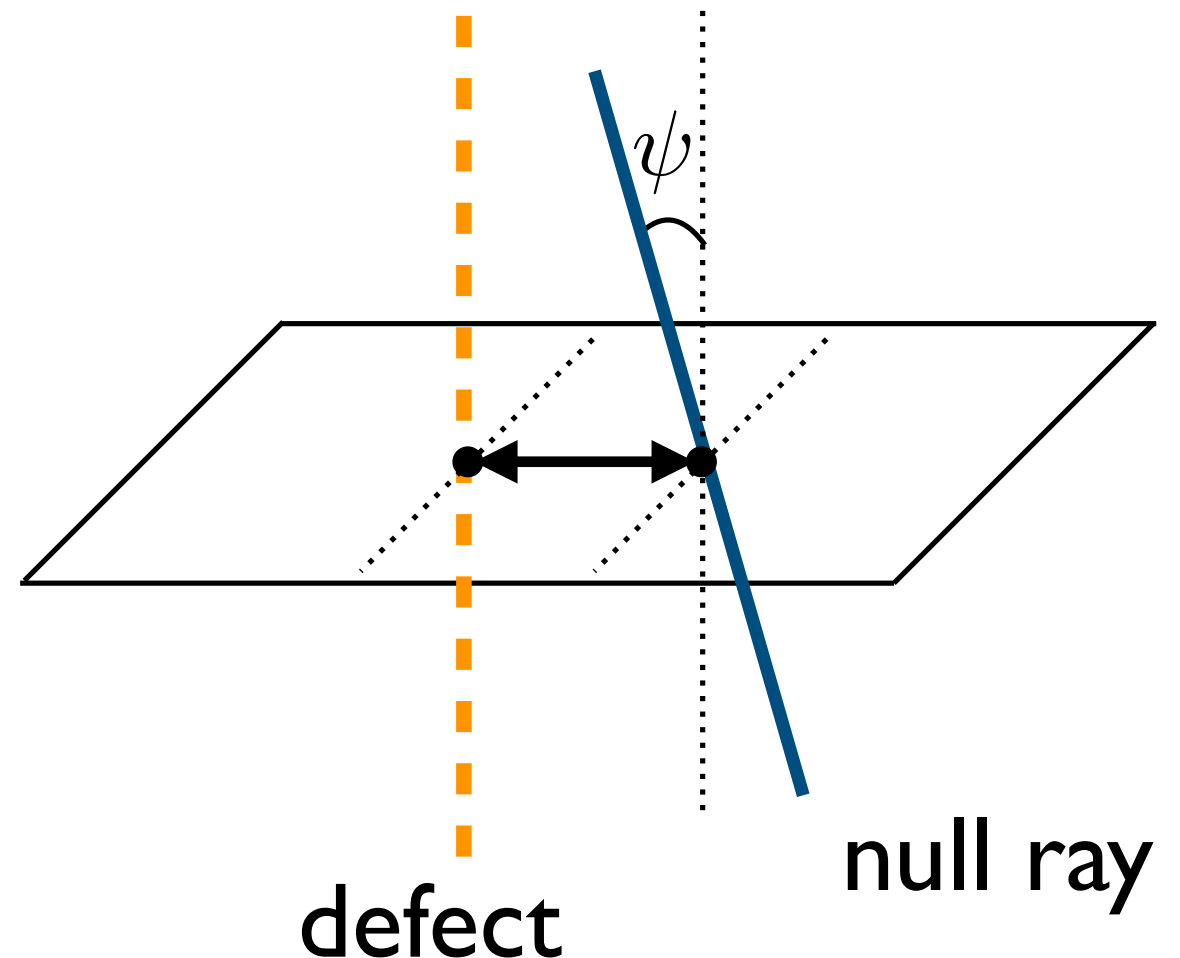
$$[T_{\mu}^{\mu}]_{2d} = \frac{b}{24\pi} \hat{R} + \frac{d_1}{24\pi} (K_{ab}^{\mu} K_{\mu}^{ab} - \frac{1}{2} K^{\mu} K_{\mu}) - \frac{d_2}{24\pi} W^{ab}_{ab}$$

Average Null Energy Condition (ANEC)

Jensen, O'Bannon, Robinson, Rodgers 1812.08745

$$\int_{-\infty}^{\infty} du \langle T_{uu} \rangle \geq 0$$

$$d_2 \geq 0$$



Central Charge

$$[T_{\mu}^{\mu}]_{2d} = \frac{b}{24\pi} \hat{R} + \frac{d_1}{24\pi} (K_{ab}^{\mu} K_{\mu}^{ab} - \frac{1}{2} K^{\mu} K_{\mu}) - \frac{d_2}{24\pi} W^{ab}_{ab}$$

$d = 2$ $\mathcal{N} = (2, 0)$ supersymmetry

$$d_1 = d_2$$

Bianchi and Lemos 1911.05082

proven for $d = 4$

conjectured for $d > 4$

Central Charge

$d = 2$ boundary or defect in $d \geq 3$ BCFT or DCFT

~~Virasoro algebra $\Rightarrow$ central charge $\mathcal{C}$~~

Stress tensor 2-pt. function

??????

Thermodynamic entropy

??????

Trace Anomaly

??????

Entanglement Entropy

??????

Central Charge

Entanglement Entropy

$$d = 2 \text{ CFT}$$

Holzhey, Larsen, Wilczek hep-th/9403108 Calabrese + Cardy hep-th/0405152

interval of length ℓ

short-distance cutoff ε

$$S_{\text{EE}} = \frac{c}{3} \ln \frac{\ell}{\varepsilon} + \dots$$

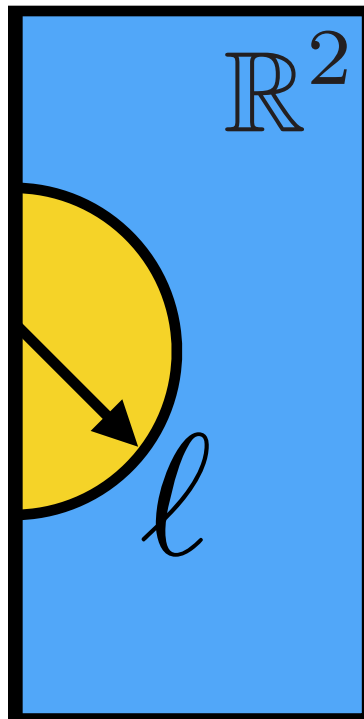
$$3 \ell \frac{d}{d\ell} S_{\text{EE}} = c$$

Central Charge

Entanglement Entropy

$$d = 3 \text{ BCFT}$$

semi-circle of radius ℓ
centered on the boundary



Central Charge

Entanglement Entropy

$$d = 3 \text{ BCFT}$$

Fursaev and Solodukhin 1305.2334, 1601.06418 Bethiere + Solodukhin 1604.07571

$$S_{\text{EE}} = \frac{1}{2} S_{\text{EE}}^{\text{CFT}} + S_{\text{EE}}^{2d}$$

$$S_{\text{EE}}^{\text{CFT}} = \# \frac{\text{Length}}{\varepsilon} + \# + \dots$$

$$S_{\text{EE}}^{2d} = \frac{b}{3} \ln(\ell/\varepsilon) + \# + \dots$$

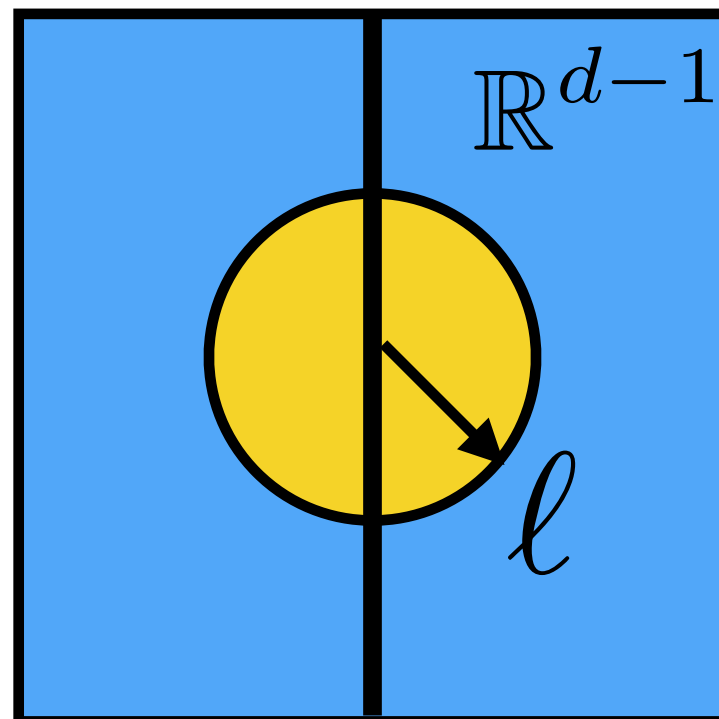
$$3 \ell \frac{d}{d\ell} \left[S_{\text{EE}} - \frac{1}{2} S_{\text{EE}}^{\text{CFT}} \right] = b$$

Central Charge

Entanglement Entropy

$$d \geq 3 \text{ DCFT}$$

sphere of radius ℓ
centered on the defect



Central Charge

Entanglement Entropy

$d \geq 3$ DCFT

Kobayashi, Nishioka, Sato, Watanabe 1810.06995
Jensen, O'Bannon, Robinson, Rodgers 1812.08745

$$S_{\text{EE}} = S_{\text{EE}}^{\text{CFT}} + S_{\text{EE}}^{2d}$$

$$S_{\text{EE}}^{\text{CFT}} = \# \frac{\text{Area}}{\varepsilon^{d-2}} + \frac{\#}{\varepsilon^{d-4}} + \dots + \# a \ln(\ell/\varepsilon) + \# + \dots$$

$$S_{\text{EE}}^{2d} = \frac{1}{3} \left(b - \frac{d-3}{d-1} d_2 \right) \ln(\ell/\varepsilon) + \# + \dots$$

$$3 \ell \frac{d}{d\ell} [S_{\text{EE}} - S_{\text{EE}}^{\text{CFT}}] = b - \frac{d-3}{d-1} d_2$$

Summary

$d = 2$ boundary or defect in $d \geq 3$ BCFT or DCFT

~~Virasoro algebra $\Rightarrow$ central charge $\mathcal{C}$~~

Stress tensor 2-pt. function

??????

Thermodynamic entropy

??????

Trace Anomaly

??????

Entanglement Entropy

??????

Outline:

- 2d CFT Central Charge
- The Systems
- Boundary/Defect Central Charges
- Examples
- Summary and Outlook

Examples

$$d = 3 \text{ BCFTs}$$

free massless real scalar or Dirac fermion

Theory	BC	b	d_1	d_2
Scalar	Dirichlet	$-1/16$	$3/32$	N/A
Scalar	Neumann	$1/16$	$3/32$	N/A
Fermion	Mixed	0	$3/16$	N/A

Nozaki, Takayanagi, Ugajin 1205.1573

Jensen and O'Bannon 1509.02160

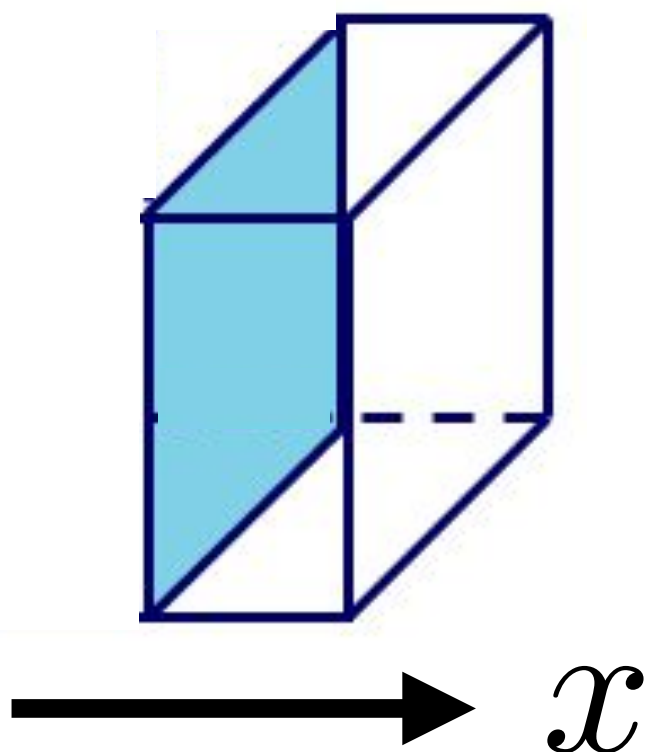
Fursaev and Solodukhin 1601.06418

Examples

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Fermion	Mixed	0	$3/16$	N/A



“mixed”

$$\Psi_{\pm} \equiv \frac{1}{2} (1 \pm \gamma^x) \Psi$$

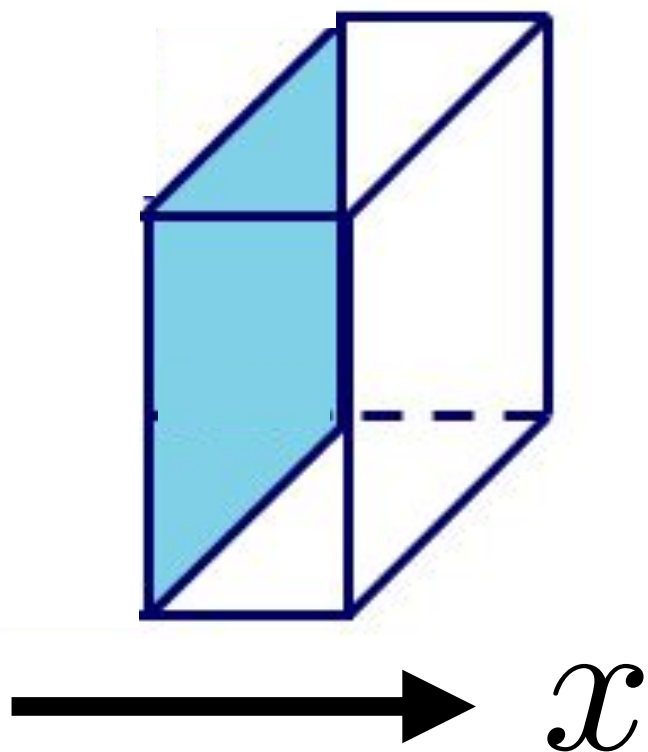
Dirichlet for Ψ_+ or Ψ_-
+ Neumann for Ψ_- or Ψ_+

Examples

$d = 3$ BCFTs

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“mixed”

$$\Psi_{\pm} \equiv \frac{1}{2} (1 \pm \gamma^x) \Psi$$

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+ Neumann for Ψ_- or Ψ_+

Examples

$d = 3$ BCFTs

free massless real scalar or Dirac fermion

Theory	BC	b	d_1	d_2
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Conjecture

Herzog, Huang, Jensen 1709.07431

$$b \geq -\frac{2}{3}d_1$$

Examples

$d = 3$ BCFTs

free massless real scalar or Dirac fermion

Theory	BC	b	d_1	d_2
Scalar	Dirichlet	$-1/16$	$3/32$	N/A
Scalar	Neumann	$1/16$	$3/32$	N/A
Fermion	Mixed	0	$3/16$	N/A

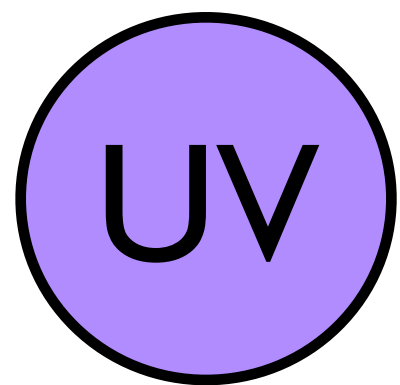
b -theorem

Jensen and O'Bannon 1509.02160

Casini, Landea, Torroba 1812.08183

RG flow on the boundary

$$b_{\text{UV}} \geq b_{\text{IR}}$$



UV BCFT

Free, massless, real scalar
Neumann B.C.

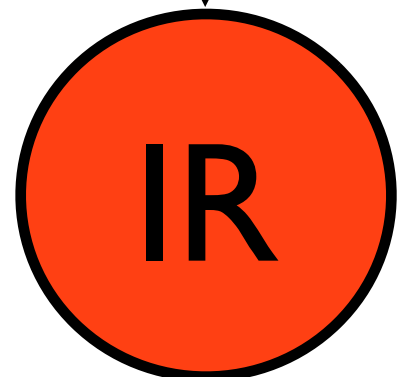
Boundary RG Flow

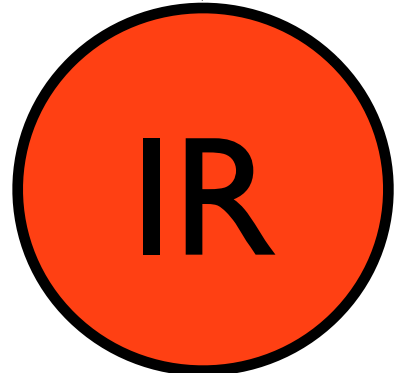
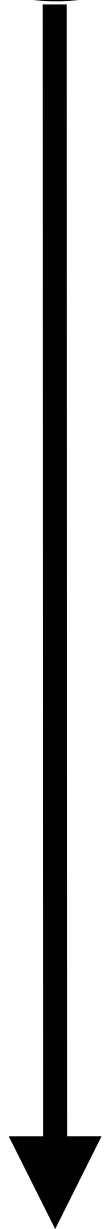
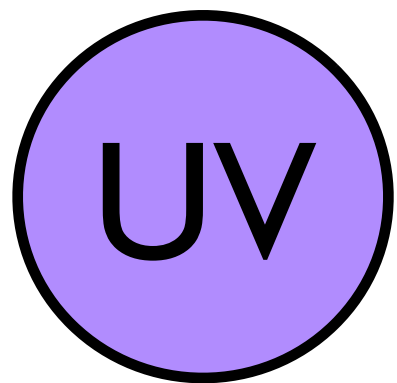
$$S_{\text{BCFT}}^{\text{UV}} \rightarrow S_{\text{BCFT}}^{\text{UV}} + \int d^3x \delta(x) m^2 \Phi^2(\vec{x})$$

$$\Delta_{\Phi^2} = 1 < 2$$

IR BCFT

Free, massless, real scalar
Dirichlet B.C.





$$b_{\text{UV}} = \frac{1}{16}$$

Neumann B.C.

$$b_{\text{UV}} \geq b_{\text{IR}}$$

$$b_{\text{IR}} = -\frac{1}{16}$$

Dirichlet B.C.

Examples

AdS_{d+1} radius of curvature L

AdS_3 probe brane

$$S_{\text{brane}} = \mathcal{T} \int d^3\xi \sqrt{\det P[G_{MN}]}$$

$$b = d_1 = d_2 = 6\pi L^3 \mathcal{T}$$

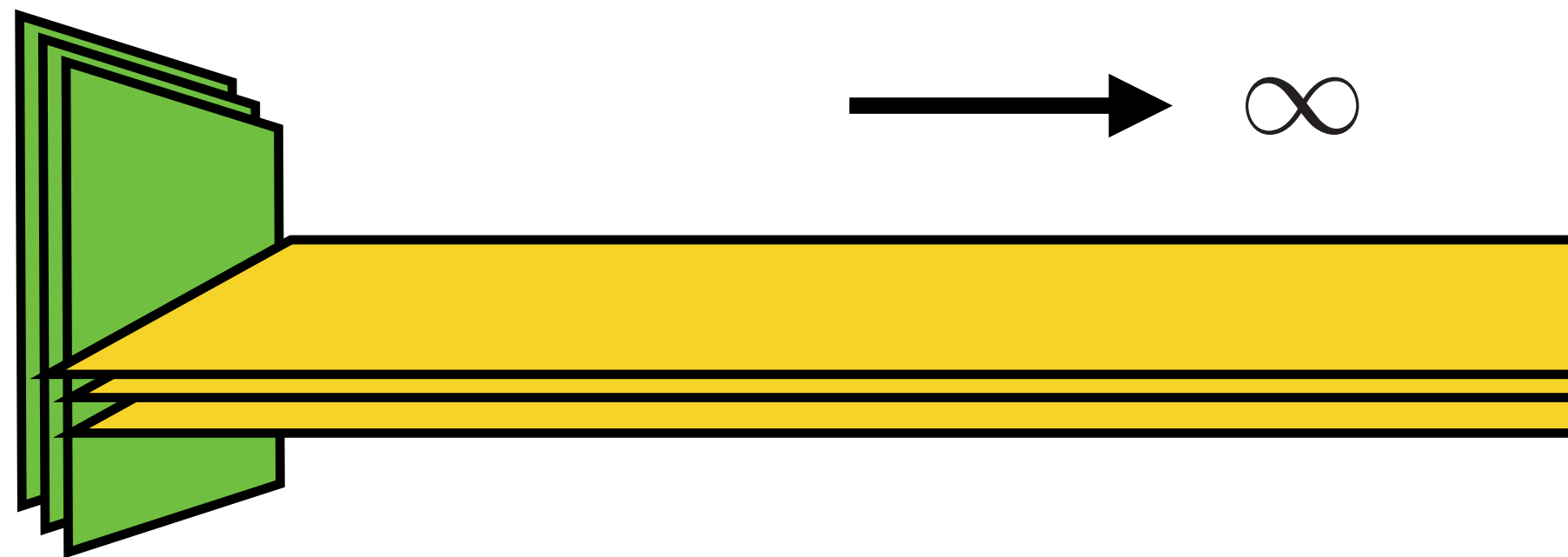
Graham + Witten hep-th/9901021

Jensen, O'Bannon, Robinson, Rodgers 1812.08745

Similar to holographic CFTs in $d = 4$, which have $a = c$

Examples

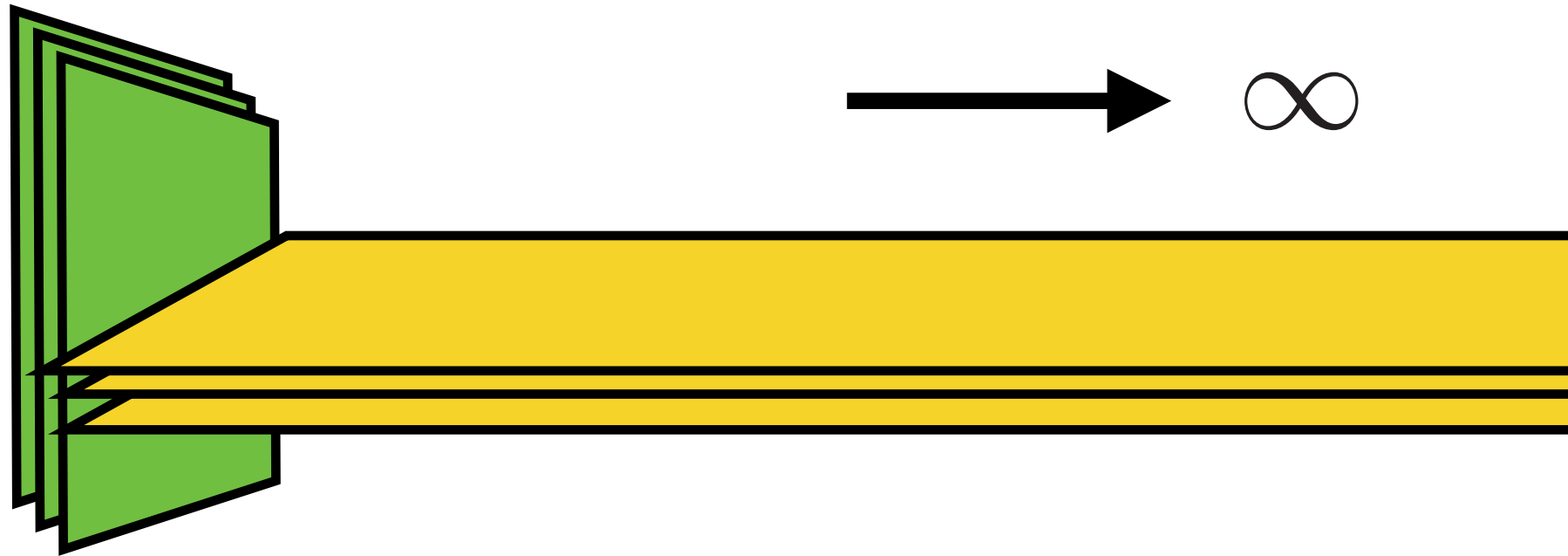
M2-branes ending on M5-branes



	0	1	2	3	4	5	6	7	8	9	10
M5	●	●	●	●	●	●	—	—	—	—	—
M2	●	●	—	—	—	—	●	—	—	—	—

Examples

M2-branes ending on M5-branes



N coincident M5-branes' worldvolume theory

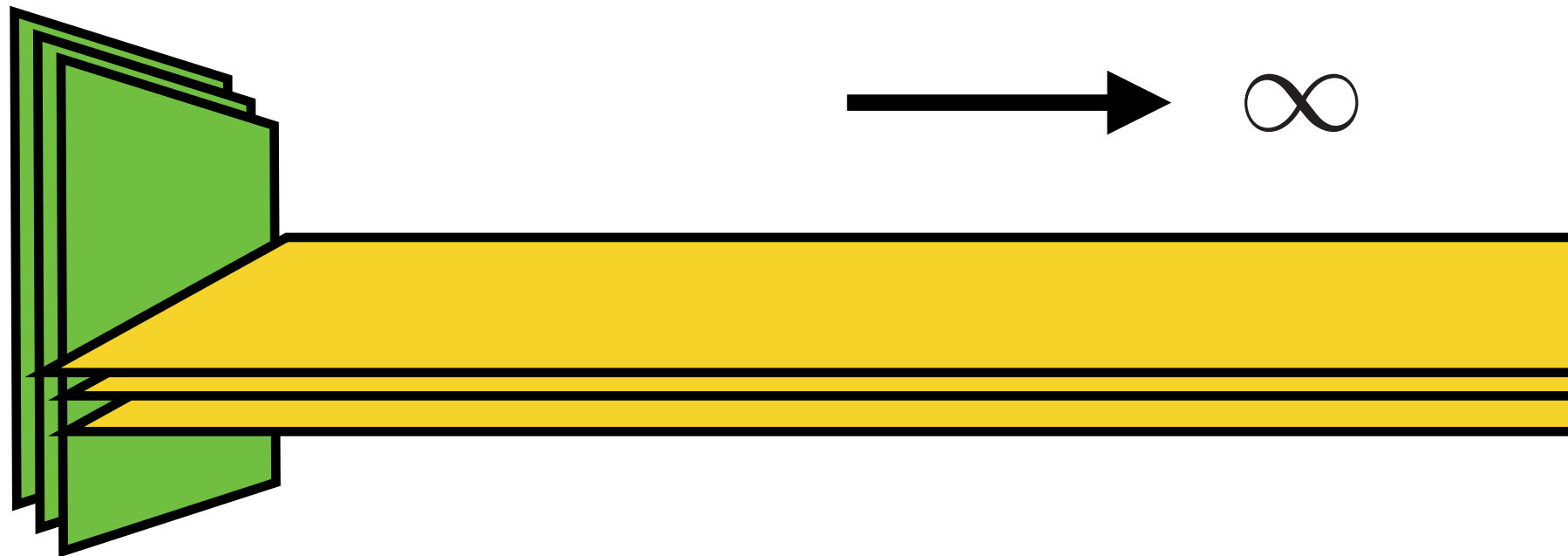
$d = 6$ $\mathcal{N} = (2, 0)$ supersymmetric CFT

Gauge algebra $U(N)$

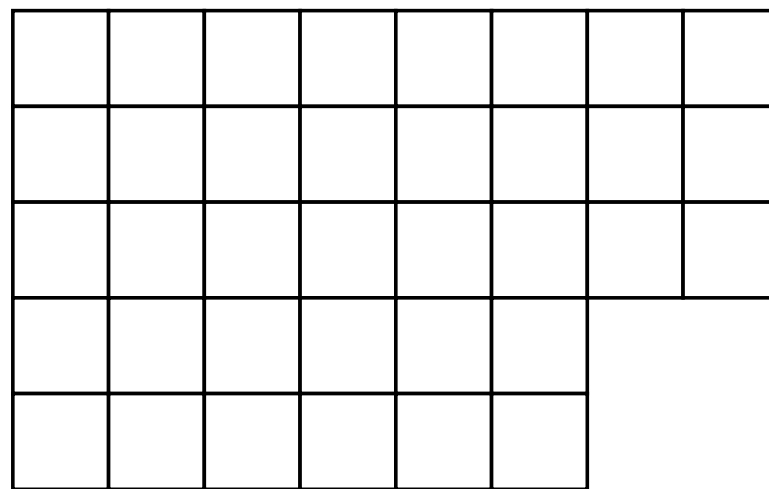
M2-branes = “Wilson surface”

Examples

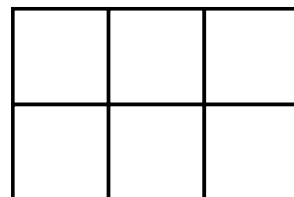
M2-branes ending on M5-branes



$\mathcal{R} \leftrightarrow$



$\vdots$



Examples

M2-branes ending on M5-branes

Estes, Krym, O'Bannon, Robinson, Rodgers 1812.00923

Jensen, O'Bannon, Robinson, Rodgers 1812.08745

Chalabi, O'Bannon, Robinson, Sisti 2003.02857

$$b = 24(\lambda, \rho) + 3(\lambda, \lambda)$$

$$d_2 = 24(\lambda, \rho) + 6(\lambda, \lambda)$$

$\lambda \equiv$ highest weight $\rho \equiv$ Weyl vector

Calculated both using holography

However a SUSY index calculation shows that this result for d_2 is exact!

Examples

M2-branes ending on M5-branes

Estes, Krym, O'Bannon, Robinson, Rodgers 1812.00923

Jensen, O'Bannon, Robinson, Rodgers 1812.08745

Chalabi, O'Bannon, Robinson, Sisti 2003.02857

$$b = 24(\lambda, \rho) + 3(\lambda, \lambda)$$

$$d_2 = 24(\lambda, \rho) + 6(\lambda, \lambda)$$

$\lambda \equiv$ highest weight $\rho \equiv$ Weyl vector

$$b \geq 0 \text{ and } d_2 \geq 0$$

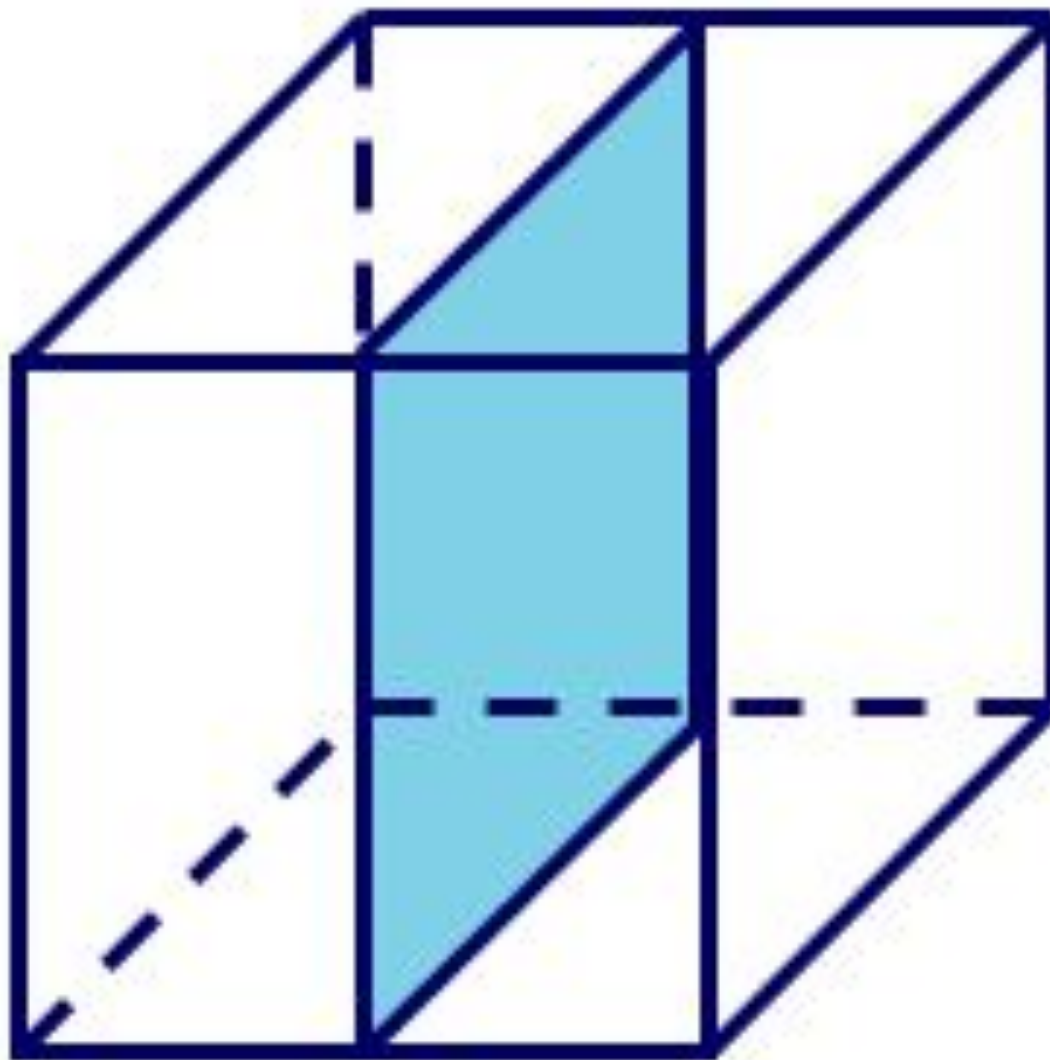
Both are invariant under the Weyl group
and complex conjugation of the representation $\mathcal{R} \rightarrow \mathcal{R}^*$

Outline:

- 2d CFT Central Charge
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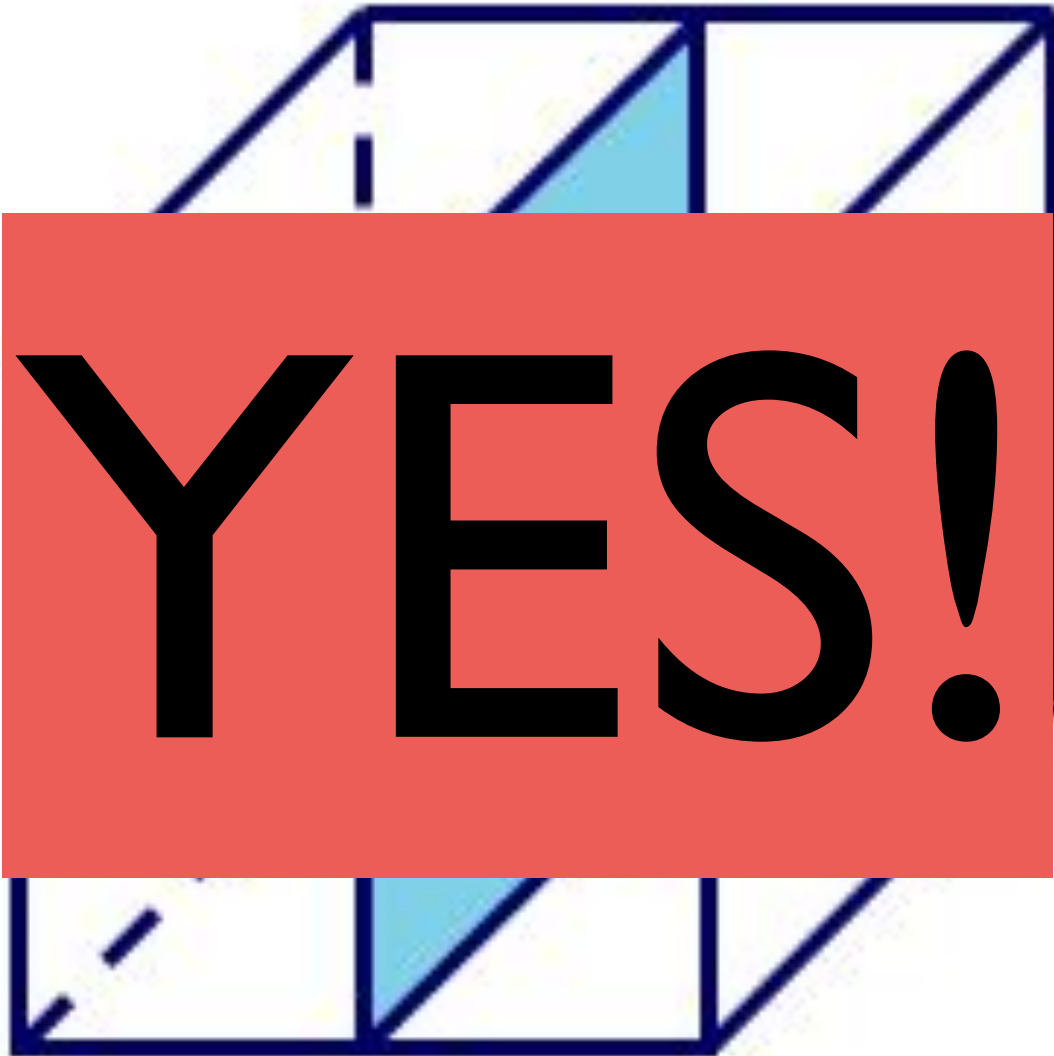
Question

Can we define a central charge
for $d = 2$ boundaries and defects?



Question

Can we define a central charge
for $d = 2$ boundaries and defects?



YES!

Summary

$d = 2$ boundary or defect in $d \geq 3$ BCFT or DCFT

~~Virasoro algebra $\Rightarrow$ central charge $\mathcal{C}$~~

Stress tensor 2-pt. function

??????

Thermodynamic entropy

??????

Trace Anomaly

??????

Entanglement Entropy

??????

Summary

$d = 2$ boundary or defect in $d \geq 3$ BCFT or DCFT

~~Virasoro algebra $\Rightarrow$ central charge $\mathcal{C}$~~

Stress tensor 2-pt. function

??????

Thermodynamic entropy

??????

Trace Anomaly



Entanglement Entropy



Outlook

More examples?

Stress-energy tensor and thermal entropy?

Constraints? Bounds? c-theorems?

Applications?

Boundaries and defects of other dimensions?

Thank You.