CFT correlators and Black Holes. Part 2

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Introduction

- HHLL correlators: computing to all orders in μ
- Application of HHLL correlators: phase shift

Common themes: holography and geometry.

1812.03120, 1904.00060, 1907.00867, 1909.05775, 2002.12254, 2005.06877 (in collaboration with R. Karlsson, M.Kulaxizi, G.S. Ng, P. Tadic)

Introduction

The two-point function of the stress-tensor in a CFT in d spacetime dimensions, $\langle T_{\mu\nu} T_{\alpha\beta} \rangle \sim C_T$ contains the central charge C_T. We will be interested in the $C_T \to \infty$ limit.

A heavy operator $\mathcal{O}_H$ will have conformal dimension $\Delta_H \sim C_T$ with $\mu \simeq \Delta_H / C_T$ fixed. If the heavy state created by $\mathcal{O}_H$ on a sphere of radius R is thermal, then in the large R limit

$$\mu \simeq \frac{\Delta_H}{C_T} \simeq \frac{\Delta_H / R}{vol(S^{d-1})} \frac{R^d}{C_T} \simeq \frac{\mathcal{E}}{C_T} R^d \simeq T^d R^d$$

where $\mathcal{E}$ is the energy density, T is the temperature.

Introduction

Consider heavy-heavy-light-light (HHLL) correlator

$$\langle \mathcal{O}_H(\infty)\mathcal{O}_L(1)\mathcal{O}_L(z,\bar{z})\mathcal{O}_H(0)\rangle$$

where $\mathcal{O}_L$ has conformal dimension $\Delta_L = \mathcal{O}(1)$. In the T-channel $(z,\bar{z}\to 1)$ conformal block decomposition includes stress-tensor sector: operators made out of stress tensor.

Stress tensor OPE fixed by Ward identity.

$$(\mathcal{O}_H \times \mathcal{O}_H - T_{\mu\nu} - \mathcal{O}_L \times \mathcal{O}_L) \sim \frac{\Delta_L}{\sqrt{C_T}}\frac{\Delta_H}{\sqrt{C_T}} \sim \mu$$

Introduction

Stress-tensor sector at $\mathcal{O}(\mu^2)$ is a result of an infinite sum over *double stress tensor operators*: $T_{\mu\nu}\partial_\alpha \dots \partial_\beta \Box^n T_{\gamma\delta}$.

$$T_{\mu\nu}\partial_{\mu_1} \dots \partial_{\mu_s} T_{\alpha\beta}, \quad \text{leading twist } \tau = 2(d-2);$$

$$T_\mu^{\ \alpha} T_{\alpha\nu}, \ T_{\mu\nu}\partial_{\mu_1} \dots \partial_{\mu_s}\Box \ T_{\alpha\beta}, \tau = 2(d-2)+2$$

$$T_{\mu\nu}T^{\mu\nu}, \ T_\mu^{\ \alpha}\Box T_{\alpha\nu}, \ T_{\mu\nu}\partial_{\mu_1} \dots \partial_{\mu_s}\Box^2 \ T_{\alpha\beta}, \tau = 2(d-2)+4$$

Leading twist k-stress tensors contribute $[\mu(1-\bar{z})^{\frac{d-2}{2}}]^k$. Will consider lightcone limit simultaneously with $\mu \to \infty$.

Introduction

The results in $d = 4$ are

$$\langle \mathcal{O}_H \mathcal{O}_H \mathcal{O}_L \mathcal{O}_L \rangle^{(1)} \simeq \mu [(1-z)(1-\bar{z})]^{-\Delta_L} (1-\bar{z}) f_3(z)$$

$$\langle \mathcal{O}_H \mathcal{O}_H \mathcal{O}_L \mathcal{O}_L \rangle^{(2)} \simeq \mu^2 \frac{(1-\bar{z})^2}{[(1-z)(1-\bar{z})]^{\Delta_L}} \left(\frac{\Delta_L}{\Delta_L - 2} \right) \times \Big[(\Delta_L - 4)(\Delta_L - 3) f_3(z)^2 + \frac{15}{7} (\Delta_L - 8) f_2(z) f_4(z) + \frac{40}{7} (\Delta_L + 1) f_1(z) f_5(z) \Big]$$

where $f_a(z) = (1-z)^a \, {}_2F_1(a, a, 2a, 1-z)$.

Introduction

One can continue this procedure and compute the stress-tensor sector to any desired order in μ. The result has the form:

$$\langle \mathcal{O}_H \mathcal{O}_H \mathcal{O}_L \mathcal{O}_L \rangle \simeq \exp\left(\Delta_L \sum_k [\mu(1-\bar{z})^{\frac{d-2}{2}}]^k \mathcal{F}^{(k)} \right)$$

In the limit $\Delta_L \to \infty$, each $\mathcal{F}^{(k)}$ has a finite limit. E.g.

$$\mathcal{F}^{(2)}_{\infty} = \frac{-5 f_3(z)^2 + \frac{15}{7} f_2(z) f_4(z) + \frac{40}{7} f_1(z) f_5(z)}{28800}$$

Introduction: comments

We can read off the OPE coefficients $\lambda_{\mathcal{O}_L \mathcal{O}_L T^k_{\mu\nu}}$ to leading order in $1/C_T$. We did not use holography, but recover the OPE coefficients which have been computed using holography (and get many more).

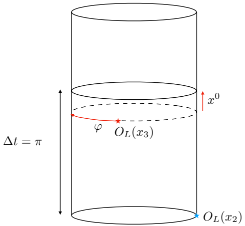
Consistent with universality of the leading twist OPE coefficients: they don't depend on higher derivative corrections in the bulk.

The closest analog of the HHLL Virasoro vacuum block.

Outline

Correlator on S^{d-1}

Correlator $\langle \mathcal{O}_H \mathcal{O}_H \mathcal{O}_L \mathcal{O}_L \rangle$ can be viewed as 2-point function $\langle \mathcal{O}_L \mathcal{O}_L \rangle_{\mathcal{O}_H}$ in the state created by $\mathcal{O}_H$ at $t = \pm\infty$.



Cross ratios $z = e^{i(\frac{\Delta t}{R} + \Delta\varphi)}$, $\bar{z} = e^{i(\frac{\Delta t}{R} - \Delta\varphi)}$.

Correlator on S^{d-1}

Will keep

$$\Delta x^- \equiv \mu^{\frac{2}{d-2}} \cdot (\Delta t/R - \Delta\varphi) \approx i\mu^{\frac{2}{d-2}}(1-\bar{z})$$

fixed as $\mu\to\infty$, $\bar{z}\to 1$. Large volume limit $(R\to\infty)$ means $\Delta x^- \sim R^{\frac{d+2}{d-2}}\to\infty$, $\Delta x^+ \sim 1/R\to 0$; $z\to 1$ [hypergeometric functions in f_a become 1]. Equivalently, only operators $T_{\mu\nu}\dots T_{\alpha\beta}$ contribute [$\Delta x^+ \equiv (\Delta t + \Delta x)/R$]

$$\mathcal{F}_\infty|_{d=4} \simeq -\log(\Delta x^+ \Delta x^-) + \frac{\Delta x^-(\Delta x^+)^3}{120} + \frac{(\Delta x^-)^2(\Delta x^+)^6}{10080} + \frac{1583(\Delta x^-)^3(\Delta x^+)^9}{648648000} + \dots,$$

Correlator on S^{d-1}

For comparison, in $d = 2$

$$\mathcal{F}|_{d=2} = \mathcal{F}_\infty|_{d=2} = -\log\sinh\left(\frac{\sqrt{\mu - 1}}{2}\Delta x^+\right).$$

which in the infinite volume limit becomes simply

$$\mathcal{F}_\infty|_{d=2} \simeq -\log\sinh\left(\frac{\sqrt{\mu}}{2}\Delta x^+\right).$$

There is no dependence on x^- and all operators have twist zero. No analog of the Virasoro block in $d > 2$ was known.

Holography

Asymptotically AdS$_{d+1}$ black hole of mass M.

$$ds^2 = -f\, dt^2 + f^{-1} dr^2 + r^2 d\Omega_{d-1}^2$$

where

$$f = 1 + \frac{r^2}{R^2} - \frac{\mu R^{d-2}}{r^{d-2}}$$

and

$$\mu \simeq \frac{G_N M}{R^{d-2}} = \frac{\ell_P^{d-1} M}{R^{d-2}}$$

$\mu \approx (R_s/R)^{d-2}$ for $\mu \ll 1$ and $\mu \approx (R_s/R)^d \simeq T^d R^d$ for $\mu \gg 1$.

Holography

Rescale coordinates as $x^- = (t-\varphi)\mu^{\frac{2}{d-2}}$ and $y = r\mu^{-\frac{1}{d-2}}$ and consider the $\mu\to\infty$ limit keeping $x^\pm$, y fixed (here $R=1$)

$$ds^2 = -\frac{1}{4}\left(1-\frac{1}{y^{d-2}}\right)(dx^+)^2 - y^2 dx^+\, dx^- + \frac{dy^2}{y^2}.$$

Two Killing vectors, ∂_+ and ∂_- give rise to two conserved quantities, K and K_+. Geodesic equation (spacelike) becomes

$$\dot{y}^2 + 4KK_+ + (y^{-2} - y^{-d})K^2 - y^2 = 0.$$

Holography

Convenient to take the large volume limit – planar horizon.

$$\Delta x^{+} \simeq 4K \int_{y_0}^{\infty} \frac{dy}{y^2 \left(y^{-d} K^2 - 4KK_{+} + y^2\right)^{\frac{1}{2}}},$$

$$\Delta x^{-} \simeq 2 \int_{y_0}^{\infty} dy \frac{-2K_{+} + y^{-d} K}{y^2 \left(y^{-d} K^2 - 4KK_{+} + y^2\right)^{\frac{1}{2}}},$$

$$\ell_{\Lambda} \simeq 2 \int_{y_0}^{\Lambda} \frac{dy}{\left(y^{-d} K^2 - 4KK_{+} + y^2\right)^{\frac{1}{2}}}.$$

$\mathcal{F}_{\infty} = -\ell$ but it's UV-divergent, need to regularize.

Holography

Define

$$\frac{I_+(x)}{4}=\int_{u_0}^{\infty}\frac{du}{u^{\frac{4-d}{2}}\,(u^{d+2}-4u^d+x)^{\frac{1}{2}}},\quad \frac{I_-(x)}{4}=\int_{u_0}^{\infty}\frac{\left(1-\frac{x}{2u^d}\right)du}{u^{\frac{4-d}{2}}\,(u^{d+2}-4u^d+x)^{\frac{1}{2}}}$$

Then the solution for the length is

$$\ell_f \simeq \log(\Delta x^+\Delta x^-) - \log[I_+(\alpha)I_-(\alpha)] + I_\ell(\alpha)$$

where α is determined by

$$(-\Delta x^-)^{\frac{d-2}{2}}(\Delta x^+)^{\frac{d+2}{2}} = \alpha\, I_-^{\frac{d-2}{2}}(\alpha)\, I_+^{\frac{d+2}{2}}(\alpha)$$

and

$$I_\ell(x)=2\int_{u_0}^{\Lambda_u}\frac{u^{\frac{d}{2}}\,du}{(u^{d+2}-4u^d+x)^{\frac{1}{2}}} - 2\log\Lambda_u$$

Holography

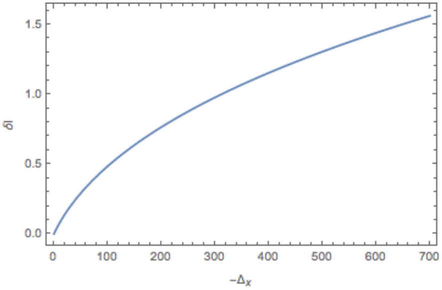
$d = 2$: recover HHLL Virasoro block:

$$-\ell_f\big|_{d=2} \simeq -\log\sinh\frac{\sqrt{\mu}\Delta x^+}{2},$$

$d = 4$: new result; expansion in $\Delta_x \equiv \Delta x^-(\Delta x^+)^3$ agrees with known data:

$$-\ell_f\big|_{d=4} \simeq -\log(\Delta x^-\Delta x^+) + \frac{\Delta_x}{120} + \frac{\Delta_x^2}{10080} + \frac{1583\Delta_x^3}{648648000} + \frac{3975313\Delta_x^4}{49401031680000} + \ldots.$$

Holography



$\mathcal{F}_\infty|_{d=4}$ with $\log(\Delta x^+ \Delta x^-)$ term subtracted as a function of $\Delta x^-(\Delta x^+)^3$.

Holography: comments.

Note that the expansions above are wrt $T^d(\Delta\tilde{x}^-)^{\frac{d-2}{2}}\Delta x^{+\frac{d+2}{2}}$.

Generally, the full correlator contains contributions other than the stress-tensor sector. In holographic CFTs, these are multi-trace operators including $\mathcal{O}_L$. In $d=2$ they all decouple: finite temperature correlator in the large volume limit is exactly the (limit of) HHLL Virasoro block.

In $d>2$ such operators might survive, but decouple in the $\Delta_L \gg 1$ limit.

Motivation

What is the stress tensor sector of an HHLL correlator good for? Are there observables which are not sensitive to other contributions (at least in holographic CFTs)?

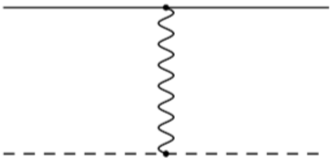
One such observable: phase shift.

Motivation

High energy gravitational scattering amplitude (large s, finite t) is described by the eikonal phase (a.k.a. the phase shift)

$$\mathcal{A} \sim e^{i\delta(s,b)}$$

where b is the impact parameter. This is the result of the summation of infinite number of diagrams, but comes from the exponentiation of



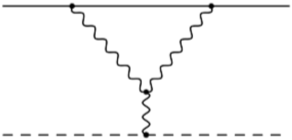
Motivation

Consider light particle of mass m and energy E scattering off a heavy particle of mass M. Assume $M \gg E \gg m$.

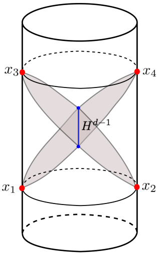
$$\delta^{(1)} = Eb \left(\frac{R_s}{b}\right)^{D-4}, \qquad R_s^{D-3} \sim G_N M$$

Higher order terms can in principle be computed. R/b is the expansion parameter. E.g.

$$\delta^{(2)} = Eb \left(\frac{R_s}{b}\right)^{2D-6}$$



Motivation

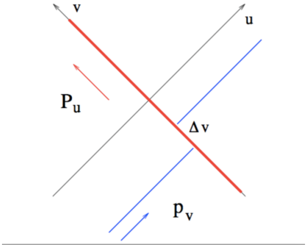


Similar expressions exist for scattering in AdS$_{d+1}$ $(D = d + 1)$:

$$e^{i\delta(p_3,p_4)} \sim \int_{x_3} \int_{x_4} e^{ip_3 x_3 + ip_4 x_4} \langle \mathcal{O}(x_1)\mathcal{O}(x_2)\mathcal{O}(x_3)\mathcal{O}(x_4)\rangle$$

Motivation

The phase shift can also be obtained by studying the propagation of a null geodesic (1st particle) in a shock wave background created by the 2nd particle.



$$\delta(s, x_\perp) = -P_1 \cdot \Delta x \simeq s\Pi(x_\perp)$$

where $\Pi(x_\perp)$ is a propagator in the transverse space.

Motivation

HHLL case: consider

$$e^{i\delta(p)} \sim \int_x e^{ipx} \langle \mathcal{O}_H(t=\infty)\mathcal{O}_L(x)\mathcal{O}_L(0)\mathcal{O}_H(t=-\infty)\rangle$$

at large momenta, the correlator is dominated by a null geodesic and $\delta(p) \approx -p \cdot \Delta x$.

Phase shift

Killing vectors ∂_t and ∂_φ (isometries w.r.t. time translation and rotation of S^{d-1}) give rise to conserved quantities p^t and p^φ:

$$p^t = \left(1 + \frac{r^2}{R^2} - \frac{\mu R^{d-2}}{r^{d-2}}\right) \frac{\partial t}{\partial \lambda}, \qquad p^\varphi = r^2 \frac{\partial \varphi}{\partial \lambda},$$

where λ is an affine parameter. Null geodesics are labeled by the impact parameter L;

$$e^{2L} = \frac{p^t + p^\varphi}{p^t - p^\varphi} = \frac{p^+}{p^-}$$

Minkowski results are recovered in the flat space limit: $R_s/R \ll 1$ ($\mu \ll 1$) and $L \approx b/R \ll 1$.

Phase shift from gravity

Null geodesic equation $g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu = 0$ (derivatives w.r.t. affine parameter λ) takes the form

$$\frac{1}{2}(\dot{r})^2 + V_{eff}(r) = \frac{1}{2}(p^t)^2$$

where

$$V_{eff}(r) = \frac{(p^\varphi)^2}{2}\left(1 + \frac{1}{r^2} - \frac{\mu}{r^d}\right)$$

One-dimensional motion. Can solve for $\dot{r}$, $\dot{t} \sim p^t$ and $\dot{\varphi} \sim p^\varphi$.

Phase shift from gravity

For $\mu = 0$ all null geodesic reimerge at the same point $t = \pi, \varphi = \pi$. Deviation from it parameterized by $\Delta x(L) = (\Delta t, \Delta\varphi)$.

$$\Delta t = 2\int_{r_0}^{\infty} \frac{\dot{t}}{\dot{r}} dr - \pi, \quad \Delta\varphi = 2\int_{r_0}^{\infty} \frac{\dot{\varphi}}{\dot{r}} dr - \pi$$

These can be computed order by order in μ.

Phase shift from gravity

The resulting phase shift $\delta = -p^\mu \Delta x_\mu$ can be expanded

$$\delta = \sum_{k=1}^{\infty} \delta^{(k)} \mu^k$$

We computed all $\delta^{(k)} = \sqrt{-p^2}\ \tilde{\delta}^{(k)}(L)$. For example,

$$\delta^{(1)} \simeq \sqrt{-p^2}\ e^{-(d-1)L}\ \ {}_2F_1(\frac{d}{2} - 1, d-2, \frac{d}{2} + 1, e^{-2L})$$

In the small impact parameter regime $L \ll 1$ we recover the Minkowski result. The radius of convergence of the series corresponds to the null geodesic approaching the circular null orbit.

CFT in $d = 2$

In the $d = 2$ case the stress-tensor sector of the correlator is known to all orders in μ (Virasoro vacuum block):

$$\langle \mathcal{O}_H(0)\mathcal{O}_H(\infty)\mathcal{O}_L(1)\mathcal{O}_L(z)\rangle \simeq \frac{1}{\left(\sin\left[\bar{\alpha}\pi + \frac{\bar{\alpha}}{2}(\Delta t \pm \varphi)\right]\right)^{\Delta_L}}$$

where $\bar{\alpha} = \sqrt{1-\mu}$. The integral picks up the pole where the argument of the sin vanishes and the phase shift is simply

$$\delta = \frac{1}{2}p^-(\Delta t + \varphi) = \pi\sqrt{-p^2}e^{-L}\left(\frac{1}{\sqrt{1-\mu}} - 1\right)$$

which agrees exactly with the gravity result.

Phase shift from CFT

$d = 2$ case provides a good illustration that the phase shift is only sensitive to the stress tensor sector (not to multi trace operators like $\mathcal{O}_L \partial^l \mathcal{O}_L$).

The full correlator of course contains such contributions, but the Fourier transform kills them. This is similar to what happens in the LLLL case.

Can't reproduce the correlator from the phase shift, but can compute the phase shift from the Regge limit of the correlator: $1 - z = \sigma e^{\rho}, 1 - \bar{z} = \sigma e^{-\rho}$; $\sigma \to 0$ with ρ fixed.

Phase shift from CFT

Generally, the Fourier transform looks like

$$\int_x e^{ipx} \langle \mathcal{O}_H \mathcal{O}_H \mathcal{O}_L \mathcal{O}_L \rangle \simeq (-p^2)^{\Delta_L - 2} \left(1 + \mu \delta^{(1)} + \dots\right)$$

where we expect $\delta^{(k)} \sim \sqrt{-p^2}$ from gravity. Inverting the $\mathcal{O}(\mu)$ term:

$$\langle \mathcal{O}_H \mathcal{O}_H \mathcal{O}_L \mathcal{O}_L \rangle^{(1)} \simeq i \int_p e^{-ipx} (-p^2)^{\Delta_L - 2} \delta^{(1)}(p) \sim \frac{i}{\sigma}$$

Note that $1/\sigma$ is consistent with the $1/\sigma^{J-1}$ behavior of the $T_{\mu\nu}$ conformal block ($J = 2$) in the Regge limit. $\delta^{(1)}$ **is reproduced from the** $T_{\mu\nu}$ **exchange exactly.**

Phase shift from CFT

At $\mathcal{O}(\mu^2)$ we encounter

$$\frac{"(\delta^{(1)})^2"}{\sigma^2}, \qquad \frac{i"\delta^{(2)}"}{\sigma}, \qquad \frac{1}{\sigma}$$

Quotation means comes from the Fourier transform of the corresponding term. $1/\sigma$ term can't compute from gravity: $\delta^{(1)} \sim \sqrt{-p^2} + \mathcal{O}(1)$.

Note that $1/\sigma^2$ naively corresponds to the contribution of spin-3 operator, but in reality it comes from the sum of all double-stress tensor operators, labeled by their spin.

Phase shift from CFT

The cross-channel sum produces

$$\langle \mathcal{O}_H \mathcal{O}_H \mathcal{O}_L \mathcal{O}_L \rangle^{(1)} \simeq i \sum_{n,\ell} \gamma^{(1)}(n,\ell) \; P^{MFT\ OPE}_{n,\ell} \; g_{n,\ell}(z,\bar{z})$$

$$\simeq i \int d\ell \int dn \int_p e^{-ipx} (-p^2)^{\Delta_L - 2} \delta\left(p^+ - n - \ell\right) \delta\left(p^- - n\right) \gamma^{(1)}(n,\ell)$$

Hence, $\delta^{(1)} = -\gamma^{(1)}(n,\ell)$ with $n,\ell \gg 1$ and $\ell/n = e^{2L} - 1$.

Regge limit: $-p^2 \gg 1$; L finite. Note that $L \to 0$ is the flat space limit and $L \to \infty$ is the limit where leading twist operators dominate.

Phase shift from CFT

Similar analysis can be done at higher orders in μ. E.g.

$$\delta^{(2)} \simeq -\gamma^{(2)} + \gamma^{(1)} \partial_n \gamma^{(1)}$$

In the flat space limit $\delta^{(2)} \simeq -\gamma^{(2)}$.

The anomalous dimensions can be independently computed as an energy shift in the AdS-Schwarzschild background. Or, the can be taken from the bootstrap calculation.

Phase shift from CFT

Consider $d = 4$:

$$\gamma^{(1)} = -\frac{3n^2}{\ell}$$

$$\gamma^{(2)} = -\frac{35}{4}\frac{(2\ell + n)n^3}{\ell^3} + 9\frac{n^3}{\ell^2}$$

which can be expanded in the large impact parameter regime:

$$\gamma^{(2)} \approx -\frac{17}{2}\frac{n^3}{\ell^2} - \frac{35}{4}\frac{n^4}{\ell^3} + \dots$$

which exactly matches the $\mathcal{O}(\mu^2)$ leading and subleading anomalous dimensions obtained from bootstrap.

Summary

- Studied the stress tensor sector of HHLL correlators.
- In the double scaling limit, where all minimal twist multi stress tensors contribute, obtained solution in the large volume limit.
- Phase shift in AdS is determined by the stress tensor sector. Computed to all orders in μ.
- Matched to the known HHLL correlator.

To do list

- Is there a closed form for generic HHLL correlators?
- Symmetry of the lightcone correlator?
- Can we make progress deconstructing thermal CFTs?
- Applications. Hydrodynamics, thermalization, quantum chaos...
- Inelastic scattering...

Thank you!