The intricate interplay between integrability and chaos for the dynamics of the probes of extremal black hole spacetimes

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Outline

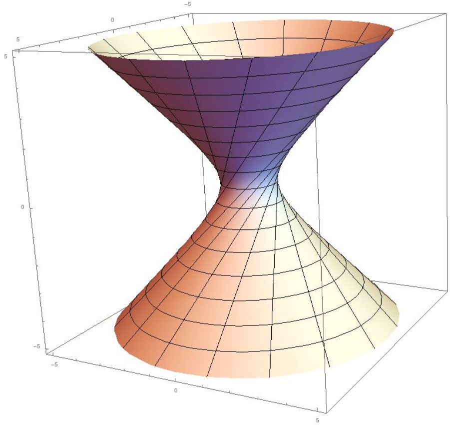
The simplest black hole

Usually the Schwarzschild black hole is considered the "simplest". This is true–for classical effects.
When quantum effects become relevant, however, it is, perhaps, the most complex–since Hawking radiation is, still, not fully understood.
The simplest black hole spacetime, where quantum effects can be consistently described, seems to be the extremal Reissner–Nordstrom black hole.
There are many ways to define it–but a geometrical way is by the statement that its "near horizon geometry" is described by the factorizable metric

$$ds^2 \approx ds^2_{\mathrm{AdS}_2} + ds^2_K$$

where, "classically", $K = S^2$. The radial-temporal part is the AdS_2, a single–sheet hyperboloid :

AdS_2, the single–sheet hyperboloid



Geometry and dynamics

- $$S_{\rm BH}=\frac{\mathcal{A}}{4}<\infty\Rightarrow N_{\rm BH}=e^{S}<\infty$$
- $$S_{\rm BH}=\ln N_{\rm BH}$$
- $$S_{\rm BH}\propto N$$

  A quantum black hole has a finite dimensional space of states. An extremal, quantum black hole, defines a space of states of fixed dimension. So the dynamics consists of unitary transformations, that keep $N_{\rm BH}$ fixed. One way is by performing operations mod N.
  So a probe of such a space must, itself, have N states. The probe must, also, preserve extremality.

From classical to quantum geometry

The classical, near–horizon, spacetime geometry, is

$$ds^2 \approx ds^2_{\mathrm{AdS}_2} + ds^2_{\mathrm{S}^2}$$

The quantum, near–horizon, spacetime geometry, when the microstates can be resolved, can be described by

$$ds^2 \approx ds^2_{\mathrm{AdS}_2[\mathrm{N}]} + ds^2_{\mathrm{S}^2_\mathrm{N}}$$

where

- $AdS_2[N]$:

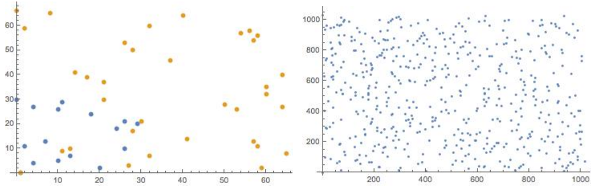
$$x_0^2 + x_1^2 - x_2^2 \equiv 1 \bmod N$$

- S_N^2 :

$$y_1^2 + y_2^2 + y_3^2 \equiv 1 \bmod N$$

Geometry mod N

- $$x_0^2 + x_1^2 - x_2^2 \equiv 1 \bmod N \Rightarrow x_0^2 + x_1^2 \equiv (1 + x_2^2) \bmod N$$
- The $x_2 = 0$ plane ($N = 31, 67, 1031$)



The cloud describes non–classical states

Classical dynamics mod N

Any point (x_0, x_1, x_2), on the hyperboloid, can be represented by a 2×2 matrix, X

$$X = \begin{pmatrix} x_0 & x_1 + x_2 \\ x_1 - x_2 & -x_0 \end{pmatrix}$$

if $\det X = -x_0^2 - x_1^2 + x_2^2 = -1$.
Moving on the hyperboloid

$$X_{n+1} = \mathrm{A} X_n \mathrm{A}^{-1} \Rightarrow X_n = \mathrm{A}^n X_0 \mathrm{A}^{-n}$$

where A in the corresponding group :

- The points, $X \in SL(2,\mathbb{R})/SO(1,1)$ in the continuum. The corresponding points mod N, belong to the group $SL(2,\mathbb{Z}_N)/SO(1,1,\mathbb{Z}_N)$.
- A particularly interesting element is the Arnol'd cat map.

The Arnol'd cat map

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} = LR^{-1}$$

where

$$L = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \qquad R = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$$

The L and R realize motion along the light cone generators of AdS_2. They satisfy the braid group relations

$$LRL = RLR$$

and

$$A^n = L^{l_1} R^{r_1} L^{l_2} R^{r_2} \cdots L^{l_n} R^{r_n}$$

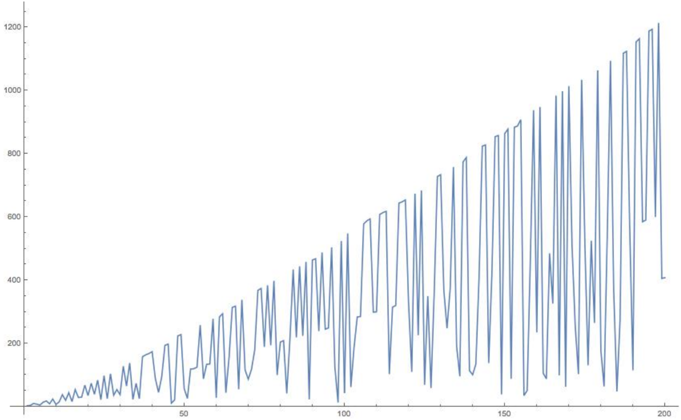
define "random walks" on the hyperboloid.

The classical dynamics of the Arnol'd cat map $\bmod N$

- $$\mathsf{A} = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$$
- $$\mathsf{A}^n = \begin{pmatrix} f_{2n-1} & f_{2n} \\ f_{2n} & f_{2n+1} \end{pmatrix}$$
- $$\mathsf{X}_n = \begin{pmatrix} f_{2n-1}f_{2n+1} + f_{2n}^2 & -2f_{2n}f_{2n-1} \\ -2f_{2n}f_{2n+1} & -f_{2n-1}f_{2n+1} - f_{2n}^2 \end{pmatrix} \bmod N$$

  where $\{f_n\}$ the Fibonacci sequence. This motion is periodic : There exists $T(N)$ s.t. $\mathsf{A}^{T(N)} \equiv I_{2\times 2} \mod N$. But $T(N)$ is a non–trivial function of N and can be much smaller than N. If $N = f_{2k}$ then $T(N) \sim \log N$.

The period $T(N)$



The quantum dynamics of the Arnol'd cat map

Quantum dynamics is described by an $N \times N$ unitary matrix, $U(\mathrm{A})_{k,l}$ given by the expression

$$U(\mathrm{A})_{k,l} = \frac{(-2|N)}{\sqrt{N}} \omega_N^{-\frac{k^2-2kl+2l^2}{2}}$$

It satisfies

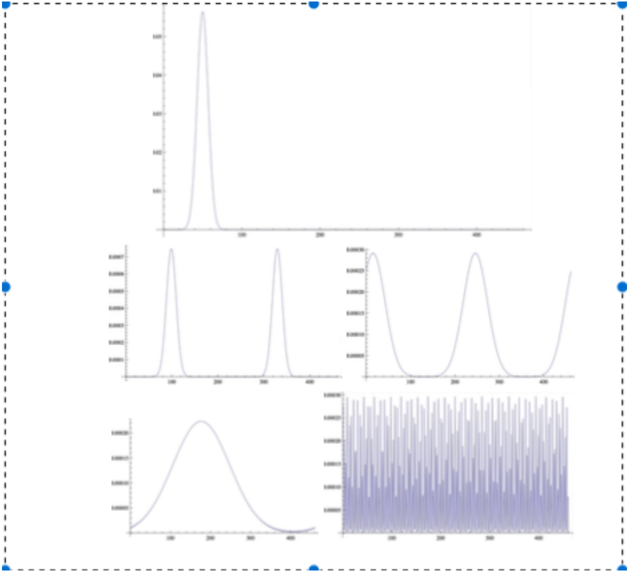
$$U(\mathrm{A})^m = U(\mathrm{A}^m)$$

This property is the key for understanding "fast scrambling" and how quantum black holes saturate the "scrambling time bound". Since A can be diagonalized

$$\mathrm{A} = \mathrm{R}\mathrm{D}\mathrm{R}^{-1}$$

an interesting basis is provided by the eigenstates of $U(\mathrm{D})$.

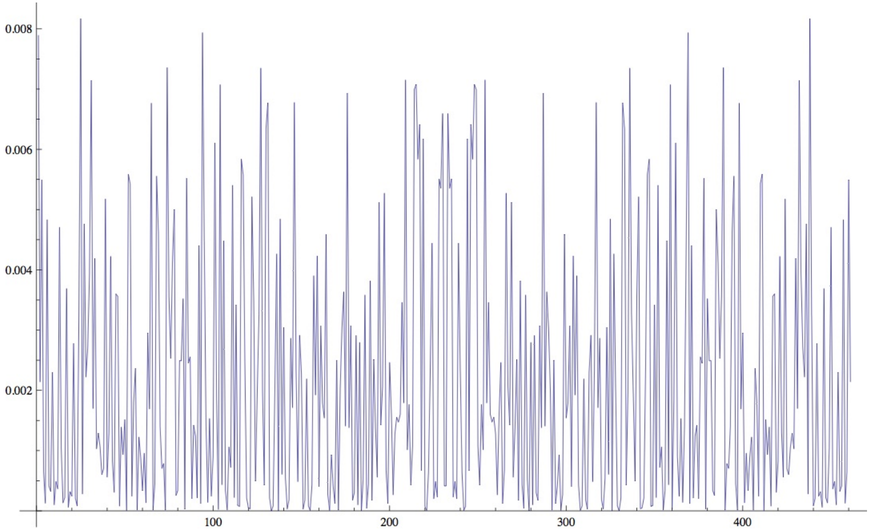
The dynamics :



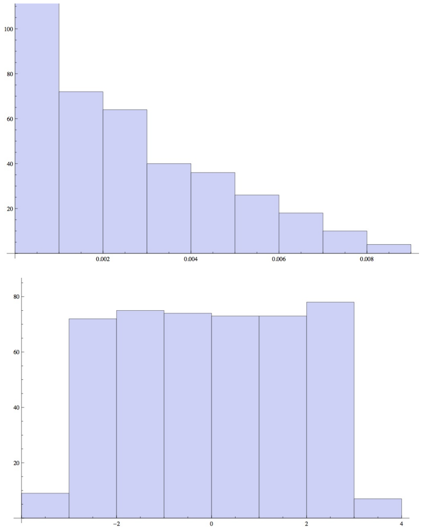
Thermodynamics and deterministic or molecular chaos

Is the probe in thermodynamic equilibrium with the black hole ? Do the black hole microstates display molecular chaos, whose macroscopic, thermodynamic, properties are described by the appropriate statistics–or do they display deterministic chaos ? Since the space of states is compact, mod N, deterministic chaos is possible, since mixing is possible.
To address these questions we study the evolution of wavepackets under $U(\mathrm{A})$ and focus on the statistical properties of their amplitude and their phase.

The ground state



The Eigenstate Thermalization Hypothesis



Conclusions and outlook

- We have a concrete example for the consistent dynamics, classical and quantum, of "point–like" probes of the spacetime of extremal black holes. We can compute transition amplitudes and probabilities from any initial to any final state of the probe, as it interacts with the black hole microstates.
- The temperature isn't that of Hawking radiation, that vanishes in the extremal limit, but of the thermodynamical description of the deterministically chaotic dynamical system of the spacetime probe of the extremal black hole. There isn't any problem with the laws of theormodynamics.
- Since the black hole is extremal, the infalling observer isn't bound to encounter the singularity. So it does make sense, in principle, to compare the measurements of two observers.