Non-Abelian gauge formalism to study the spin interactions in superconductors

non-Abelian gauge-covariant transport
Andreev-Wilson loop in ballistic Josephson junction with magnetic interactions
non-Abelian electrostatics in Josephson systems
magneto-electric φ_0-Josephson junction

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Motivations

- Topological superconductivity: requires spin-orbit $\oplus$ spin-splitting. Questions about impurities, temperature, ... $\Rightarrow$ phenomenology of the topological superconductivity ?
- Generic effect with spin-orbit coupling (SOC) in superconductors $\Rightarrow$ precursor of the topological regime / super-spintronics ?
- Beyond topological superconductivity ?

Unification of the phenomenology (temperature, impurities, magneto-electric competition, charge and spin coexistence, ...) of the spin-orbit interaction (SOC) and the spin-splitting effect (Zeeman or exchange) in superconductors ?

1. Combine SOC + Zeeman + superconductivity in a **gauge-covariant** QFT
2. Perturbative approach = **semi-classic** expansion
3. Example of DC Josephson effect + a bit of bulk

Spin-orbit as a non-Abelian gauge potential

2D system with SOC (Rashba example)

$$H = \frac{p_x^2}{2m} + \frac{p_y^2}{2m} - \mu + \boldsymbol{h}\cdot\boldsymbol{\sigma} + v_{\text{so}}(\boldsymbol{\sigma}\times\boldsymbol{p})\cdot\boldsymbol{n}$$
$$= \frac{(p_x - mv_{\text{so}}\sigma_y)^2}{2m} + \frac{(p_y + mv_{\text{so}}\sigma_x)^2}{2m} - \tilde{\mu} + A_0$$

with a (non-Abelian) gauge-potential $\boldsymbol{A} = \xi_{\text{so}}^{-1}\left(\sigma^2, -\sigma^1, 0\right)$, $\xi_{\text{so}} = \hbar/mv_{\text{so}}$: spin-orbit length, $\tilde{\mu} = \mu + mv_{\text{so}}^2/2$, $A_0 = \boldsymbol{h}\cdot\boldsymbol{\sigma}$ Zeeman interaction, ...

Spin interactions ⇔ **gauge theory**

- Rashba SOC: $A_x^y = -A_y^x = \alpha \Rightarrow F_{xy} = \alpha^2\sigma^z/2 \Rightarrow F_{xy}^z = \alpha^2$
- Dresselhaus SOC: $A_x^x = -A_y^y = -\beta \Rightarrow F_{xy} = \beta^2\sigma^z/2 \Rightarrow F_{xy}^z = \beta^2$

Gauge-covariant Wigner transform / semi-classic limit

- Perturbation theory: semi-classic expansion $\Rightarrow$ gauge-covariant ?

$$G(p,x) = \int d\mathfrak{x} \left[e^{-\mathbf{i}p\cdot\mathfrak{x}/\hbar} G\left(x+\frac{\mathfrak{x}}{2}, x-\frac{\mathfrak{x}}{2}\right)\right]$$
$$= \int d\mathfrak{x} \left[e^{-\mathbf{i}p\cdot\mathfrak{x}/\hbar} e^{\mathfrak{x}\cdot\partial/2} G(x,x)\, e^{-\mathfrak{x}\cdot\partial^{\dagger}/2}\right]$$

Gauge-covariant Wigner transform, $\partial \to D = \partial - \mathbf{i}A$

$$G(p,x) = \int d\mathfrak{x} \left[e^{-\mathbf{i}p\cdot\mathfrak{x}/\hbar} U\left(x, x+\frac{\mathfrak{x}}{2}\right) G\left(x+\frac{\mathfrak{x}}{2}, x-\frac{\mathfrak{x}}{2}\right) U\left(x-\frac{\mathfrak{x}}{2}, x\right)\right]$$

$$(\mathfrak{x}\cdot D)\, U = 0 \Rightarrow U(b,a) = \mathrm{Pexp}\left[\mathbf{i}\int_a^b dz^{\mu} A_{\mu}(z)\right]$$

parralel transport *along straight-lines*

- $A \to 0 \Rightarrow D \to \partial \Rightarrow$ usual Wigner transform

see also Gorini et al. arXiv:1003.5763

Non-Abelian gauge-covariant transport equation

$$\mathbf{i}\left(\tau_3\frac{\partial\check{G}}{\partial t_1}+\frac{\partial\check{G}}{\partial t_2}\tau_3\right)+\frac{\mathbf{i}}{m}p_k\mathfrak{D}_k\check{G}+\left[\tau_3 A_0+\check{\Delta}\left(x\right)+\frac{\mathbf{i}\left\langle\check{g}\right\rangle}{2\tau},\check{G}\left(p,x\right)\right]$$
$$-\frac{\mathbf{i}}{2}\left\{\tau_3 F_{0i}\left(x\right)+\frac{p_k}{m}F_{ki}\left(x\right),\frac{\partial\check{G}}{\partial p_i}\right\}+\frac{\mathbf{i}}{2}\left\{\frac{\partial\check{\Delta}}{\partial x_k},\frac{\partial\check{G}}{\partial p_k}\right\}\approx 0$$

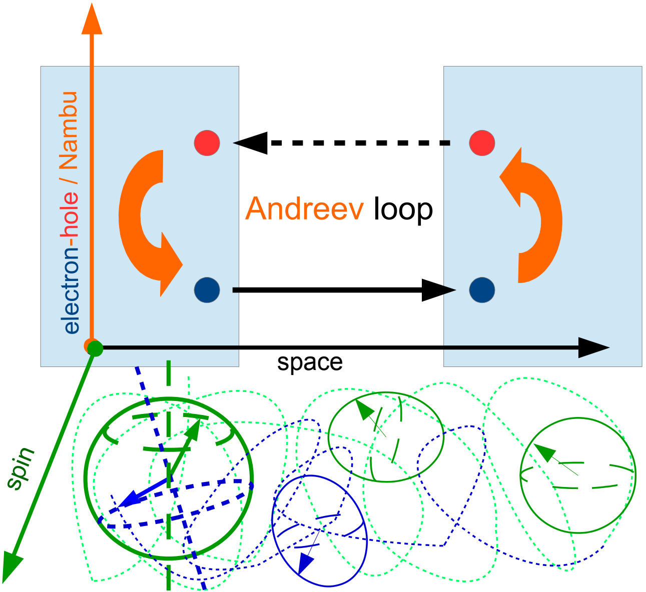
- Similar to Boltzmann equation ($\vec{F}=E+v\times B$)

$$\frac{\partial f}{\partial t}+v\frac{\partial f}{\partial x}-\vec{F}\frac{\partial f}{\partial p}=\text{collisions}$$

- Covariant derivative $\mathfrak{D}_i\check{G}=\dfrac{\partial\check{G}}{\partial x_i}-\mathbf{i}\left[\check{A}_i,\check{G}\right]$ keep tracks of the spin-orbit effect
- Gauge field $F_{\alpha\beta}=\dfrac{\partial A_\alpha}{\partial x_\beta}-\dfrac{\partial A_\beta}{\partial x_\alpha}-\mathbf{i}\left[A_\alpha,A_\beta\right]$ (non-Abelian) $\Rightarrow$ momentum + anti-commutator structure ... responsible for Hall-like physics / Lorentz force

arXiv:1403.1797 (and references therein)

Josephson effect = Andreev-Wilson loop



Holonomy

$$W = e^{\mathbf{i}(\mathfrak{n}\cdot\boldsymbol{\sigma})\Phi}$$

in the Nambu $\otimes$ spin $\otimes$ space propagation

Magnetic Josephson junction in the ballistic limit

$$g = -\frac{\mathbf{i}}{2}\sum_{s=\pm}\left(1+s\left(\mathfrak{n}\left(x\right)\cdot\boldsymbol{\sigma}\right)\right)\tan\left(\frac{EL}{v_F}+\arcsin\frac{E}{\Delta}+\frac{\varphi}{2}+s\frac{\Phi}{2}\right)$$

$$W\left(x\right) = u\left(x,x_R\right)\bar{u}\left(x_R,x_L\right)u\left(x_L,x\right) = e^{\mathbf{i}\left(\mathfrak{n}\left(x\right)\cdot\boldsymbol{\sigma}\right)\Phi}$$

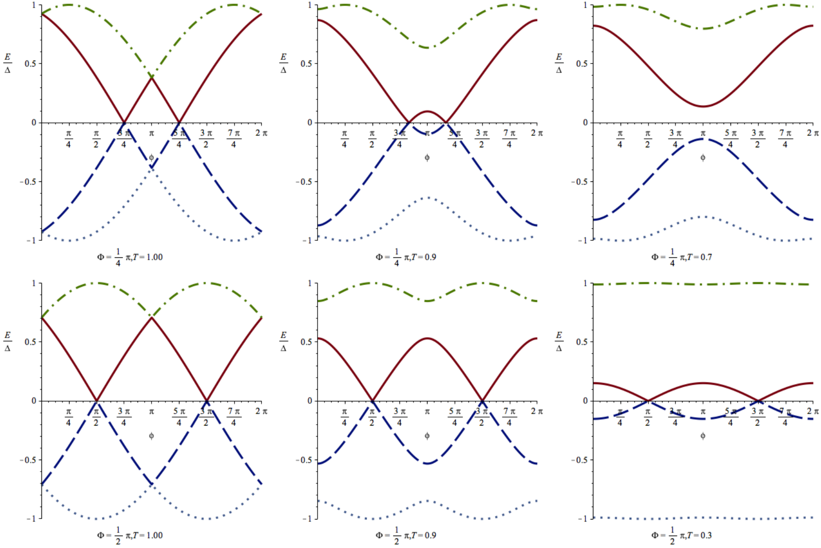
- $u\left(x_2,x_1\right) = \text{Pexp}\left\{-\mathbf{i}\int_{x_1}^{x_2}\left(\boldsymbol{B}\cdot\boldsymbol{\sigma}\right)\frac{dx}{v_F}\right\}$ transports the magnetic interactions along the junction (path-ordered exponential)
- $2\cos\Phi = \text{Tr}\left\{e^{\mathbf{i}\left(\mathfrak{n}\cdot\boldsymbol{\sigma}\right)\Phi}\right\}$; in S/F/S: $\Phi = \dfrac{2hL}{v_F}$
- Classical spin (quantization axis) precession $\left(\boldsymbol{v_F}\cdot\boldsymbol{\nabla}\right)\mathfrak{n} = 2\boldsymbol{B}\left(\boldsymbol{x}\right)\times\mathfrak{n}$
- A Josephson junction $\equiv$ **an Andreev-Wilson loop analog computer**
- Andreev bound states for the 1D-junction with one scatterer

$$\cos\left(\frac{2EL}{v_F}+s\Phi+2\arcsin\frac{E}{\Delta}\right)+T\cos\varphi+R\cos\left(\frac{2EL}{v_F}+s\Phi\right)=0$$

arXiv:1601.02973

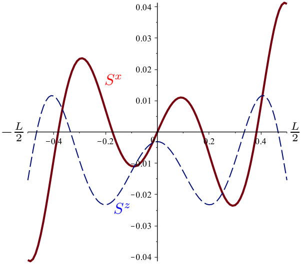
Andreev bound states in the short junction limit

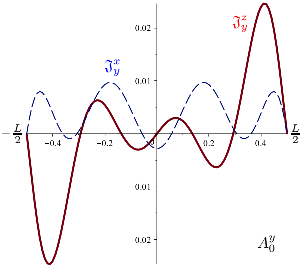
Andreev bound states for the short junction $\Phi \gg 2\omega L/v_F$



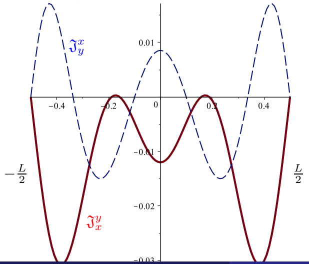
Magnetization and spin current for S/N/S with SOC

Gauß: $S = \mathfrak{D}_i F_{i0}$

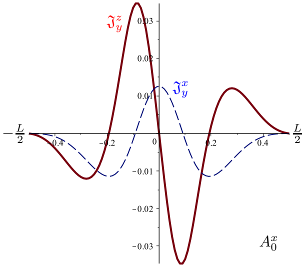




Ampère: $\mathfrak{J}_i = \mathfrak{D}_0 F_{0i}$



$\mathfrak{J}_y^{(4)} \propto \mathfrak{D}_0 \left(\mathfrak{D}_y F_{x0} + \mathfrak{D}_x F_{y0} \right)$



DC Josephson effect with spin texture + gauge-field

- Next order non-trivial example

$$\frac{\mathbf{i}}{m} p_k \mathfrak{D}_k \check{G} + \left[\tau_3 \left(\omega + A_0 \right) + \check{\Delta}\left(x \right), \check{G}\left(p, x \right) \right] + \left[\frac{\mathbf{i} \left\langle \check{g} \right\rangle}{2\tau}, \check{G}\left(p, x \right) \right] - \frac{\mathbf{i}}{2} \left\{ \tau_3 F_{0i} + \frac{p_k}{m} F_{ki}, \frac{\partial \check{G}}{\partial p_i} \right\} + \frac{\mathbf{i}}{2} \left\{ \frac{\partial \check{\Delta}}{\partial x_k}, \frac{\partial \check{G}}{\partial p_k} \right\} = 0$$

- Impurities = ballistic and diffusive limits
- Gauge field responsible for Hall-like phenomenology

OPEN QUESTION: transport equation / mixing between commutator and anti-commutators / numerical implementation ?

Edelstein and inverse Edelstein effects

In s-wave superconductors

$$j_k = \tilde{\chi}_{ki} \frac{\partial \varphi}{\partial x_i} \;\; / \;\; S^a = \tilde{\tilde{\chi}}^{ab} A_0^b$$

In s-wave superconductors + spin-orbit

- Phase gradient induces spin polarisation

$$S^a = \chi_k^a \frac{\partial \varphi}{\partial x_k} = \text{Edelstein effect}$$

- Magnetization induces supercurrent

$$j_k = e \chi_k^a A_0^a = \text{Inverse Edelstein effect}$$

  (a.k.a. spin galvanic effect)

- Ginzburg-Landau formalism: Lifshitz invariant

$$F_L \sim A_0^a \chi_k^a \frac{\partial \varphi}{\partial x_k}$$

Edelstein effects in superconductors for $T \to T_c$

Diffusive limit

$$\chi_i^a = 4\pi N_0 \frac{\tau}{2m} T \sum_{\omega_n>0} \frac{\Delta^2}{\omega_n^2} \left[\left(\hat{\Gamma} + \kappa_\omega^2 \right)^{-1} \right]^{ab} \mathfrak{J}_i^b \; ; \; \hat{\Gamma} = -\hat{P}_i \hat{P}_i , \; \hat{P}_i^{ab} = \varepsilon^{abc} A_i^c$$

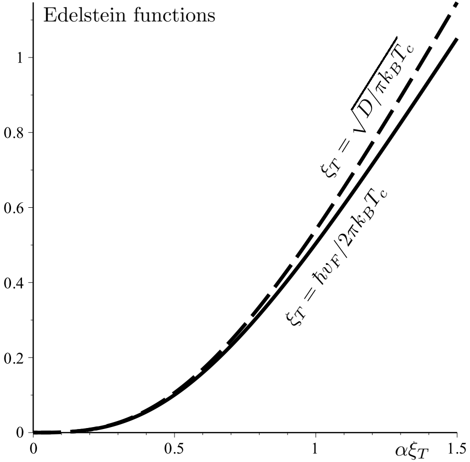
$$\text{Rashba} \Rightarrow \chi_i^a = \left(\delta_{ix}^{ay} - \delta_{iy}^{ax} \right) 4\pi N_0 \frac{D\tau}{2m} T \sum_{\omega_n>0} \frac{\Delta^2}{\omega_n^2} \frac{\alpha^3}{2\left|\omega_n\right| + D\alpha^2}$$

Ballistic limit

$$\chi_i^a = -2\pi \frac{N_0 v_F}{2m} T \sum_{\omega_n>0} \frac{\Delta^2}{\left|\omega_n\right|^3} \left\langle \left[\left(v_F n_k \hat{P}_k + 2\omega_n \right)^{-1} \right]^{ab} n_j F_{ji}^b \right\rangle$$

$$\text{Rashba} \Rightarrow \chi_i^a = \left(\delta_{ix}^{ay} - \delta_{iy}^{ax} \right) \frac{\pi N_0 \Delta^2}{4 v_F m} T \sum_{\omega_n>0} \frac{\left(v_F \alpha\right)^3}{\left|\omega_n\right|^3 \left[\left(2\omega_n\right)^2 + \left(v_F\alpha\right)^2 \right]}$$

Edelstein effects in superconductors: Edelstein functions

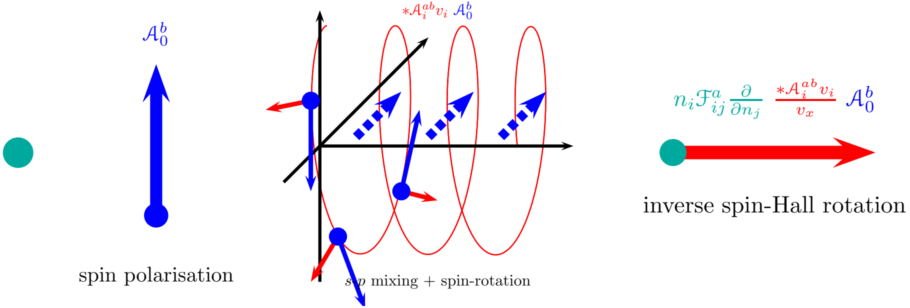


$$\sum_{\omega_n > 0} \frac{\Delta^2}{\omega_n^2} \frac{\alpha^3}{2\left|\omega_n\right| + D\alpha^2}$$

$$\sum_{\omega_n > 0} \frac{\left(v_F \alpha\right)^3}{\left|\omega_n\right|^3 \left[\left(2\omega_n\right)^2 + \left(v_F \alpha\right)^2\right]}$$

arXiv:1506.02977

Hall-like physics via the gauge-field and φ_0 effect



$$j_x\left(\boldsymbol{\varphi = 0}\right) = -\frac{e\pi N_0 v_F}{E_F}\frac{L^3}{3}\Delta^2 T_c F^a_{xi} F^a_{i0}\mathcal{O}\left(1\right) \quad ; \quad \varphi_0 \sim \frac{g\mu_B B}{E_F}\frac{L^3}{\xi_{\text{so}}^3}$$

Inverse Edelstein effect and φ_0 phase shift

$\mathrm{Tr}\left\{F_{xk}F_{k0}\right\} = \mathfrak{J}^a_x A^a_0$ inverse Edelstein effect: conversion of a spin polarisation into a charge current via spin current, responsible for the $j = j_c \sin\left(\varphi - \varphi_0\right)$ CPR

arXiv:1506.02977 + diffusive + all T + dirty interface + bulk + ...

Conclusion

- Spin-orbit interaction via gauge-covariance $\Rightarrow$ non-Abelian gauge-covariant transport formalism $\Rightarrow$ **unifying phenomenology**
- Josephson junction $\Leftrightarrow$ holonomy / Wilson loop $W(s) = e^{\mathbf{i}(\mathfrak{n}\cdot\boldsymbol{\sigma})\Phi}$
  - spin observables $\propto \mathfrak{n}$ (e.g. spin-current, spin-polarisation)
  - scalar observables $\propto \Phi$ (e.g. DOS, charge-current)
- Spin current $\mathfrak{J}_\mathfrak{i}$ and Spin polarisation S follow non-Abelian electrostatics

$$S \propto \mathfrak{D}_i F_{i0} \text{ Gauß } ; \quad \mathfrak{J}_i = \mathfrak{D}_0 F_{0i} \text{ Ampère}$$

  plus higher order exotic behaviors (non-Abelian $\mathfrak{D}_\alpha F = \partial_\alpha F - \mathbf{i}\left[A_\alpha, F\right]$)
- $j(\varphi) = j_c \sin(\varphi - \varphi_0)$ due to (inverse) spin-Hall effect, with $\varphi_0 \propto \frac{\hbar}{E_F} L^3 J_x^a A_0^a$ proportional to spin-current and spin-polarisation:
  - appear for long junction: L^2 effect in S/F/S vs. L^3 effect for φ_0
  - robust to the diffusive limit, dirty interface (allows for a shorter junction), temperature, ...
  - generalisation of the direct-/inverse-Edelstein effect in bulk superconductors irrespective of the spin-orbit strength and symmetry

Perspectives

- Complete the magneto-electric phenomenology in an effective field theory (i.e. Lifshitz invariant in a Ginzburg-Landau formalism)
- Vortex phenomenology in superconductors with spin-orbit effect ?
- Spin Hall phenomenology ?
- Understand the topological effects in the formalism (real space topology ?)
- Where is the Berry curvature ?
- Add the (full) phase-space topology
- Add the self-consistent gauge fields dynamics