A TALE OF TWO DYONS

Gérard CLÉMENT
LAPTh, Annecy-le-Vieux, France
Dmitri GAL’TSOV
MSU, Moscow, Russia
Concerning the title:

After my abstract was submitted, I discovered that Gary had published in 1997 (with Neil Cornish) a paper with an almost similar title: “The tale of two centres”.

Actually, when choosing this title I had in mind Edgar Poe’s detective tales, particularly “The gold-bug”, in which deciphering a secret message leads to a buried treasure...
4D Einstein-Maxwell: old-fashioned theory, but still full of surprises!

– I will present a “new” exact stationary, asympt. flat solution of the EM equations, and the successive clues leading to its interpretation as a complex physical system of two extreme co-rotating dyonic NUTty black holes, held apart by an electrically charged rod which also acts as a Dirac-Misner string.

– Solution constructed 20 years ago (GC1997) using finite Geroch transformation.
- Restriction of EM4 to **stationary** solutions
  (1 timelike Killing vector) \(\rightarrow\)
  reduction to 3D gravitating \(\sigma\) model
  with “hidden” symmetry group \(SU(2, 1)\)
  of transformations between solutions
  (e.g. Schwarzschild \(\rightarrow\) RN-NUT).
  **Static** and **rotating** solutions are not
  related by \(SU(2, 1)\) transformations.

- Solutions with 2 commuting Killing vectors
  (e.g. **stationary axisymmetric**):
  combination of infinitesimal \(SU(2, 1)\) transf.
  with infinitesimal linear transf. \(K_i \rightarrow K'_i\)
  in the 2-Killing vector space
  \(\rightarrow\) infinite dimensional **Geroch** group
  \(\Rightarrow\) complete integrability.
  Basis for **inverse scattering** transform methods.
Finite Geroch transformation


- **Problem**: A linear transf. $\partial_t \rightarrow \partial'_t = \alpha \partial_t + \beta \partial_\varphi$ changes the asymptotic behaviour ("centrifugal force").

- **Solution**: Combine this with an $SU(2,1)$ transformation also changing the as. behaviour.

- **Bertotti-Robinson** solution to EM4: geometry $AdS_2 \times S^2$

$$ds^2 = -(x^2 - 1)dt^2 + \frac{dx^2}{x^2 - 1} + \frac{dy^2}{1 - y^2} + (1 - y^2)d\varphi^2$$

generated by a constant electric field $A = xdt$.

- This is related to Schwarzschild by an $SU(2,1)$ transf. $\Pi : S \rightarrow BR$
– The linear transformation $\mathcal{R}(\Omega, \gamma)$:
\[ d\varphi = d\varphi' - \Omega dt', \quad dt = \gamma dt' \]
acting on $\text{BR}$ does not change the as. behaviour ($x \to \infty$).

– The combined transformation
\[ \Sigma = \Pi^{-1} \mathcal{R}(\Omega, \gamma) \Pi \]
transforms Schwarzschild $\to$ Kerr.

– More generally, $\Sigma$ acting on as. flat monopole stationary axisymm. solution of EM $\to$ as. flat monopole + dipole solution.
• Axisymmetric stationary metric (Weyl):
\[ ds^2 = -F(dt - \omega d\varphi)^2 + F^{-1}[e^{2\kappa}(d\rho^2 + dz^2) + \rho^2 d\varphi^2]. \]
Prolate spheroidal coords. \((x, y)\):
\[ \rho = \kappa(x^2 - 1)^{1/2}(1 - y^2)^{1/2}, \quad z = \kappa xy. \]
– Zipoy1966-Voorhees1970 vacuum metric:
\[ F = \left(\frac{x - 1}{x + 1}\right)^{\delta}, \quad e^{2\kappa} = \left(\frac{x^2 - 1}{x^2 - y^2}\right)^{\delta^2}, \quad \omega = 0 \quad (\delta \in \mathbb{R}) \]
naked curvature singularity at \(x = 1\),
except for \(\delta = 1\) (Schwarzschild);
\(\delta = 2\) solution first given by Darmois1927.

Rotating generalization: Tomimatsu+Sato1972
(\(\delta\) integer). \(\delta = 2\): naked ring singularity;
– Gibbons+Russel-Clark1973: TS2 has a causal boundary \((g_{\varphi\varphi} = 0)\) and a non-curvature
Misner-string singularity at \(x = 1\);
– Kodama+Hikida2003: Two degenerate
horizons at \(x = \pm y = 1\), and a conical singularity at \(x = 1\).
• Transformation $\Sigma$ acting on $Z\nu$ → continuous family of rotating spacetimes with dipole electromagnetic field.

– $\delta = 1$: Kerr; $\delta = 1 + \epsilon$: “almost Kerr”.

– Rotating solution for $\delta = 2$: Ernst potentials:
  \[ \mathcal{E} = \frac{U - W}{U + W}, \quad \psi = \frac{V}{U + W}. \]

  Kinnersley potentials:

  \[ U = p \frac{x^2 + 1}{2x} - i q y, \quad (p = \sqrt{1 - q^2}, \; \epsilon^2 = 1) \]

  \[ W = 1 + \frac{q^2 1 - y^2}{2 x^2 - 1} + i \frac{pqy}{2x}, \quad V = \epsilon(1 - W). \]

– Simple enough, but for the explicit solution, must dualize (recover the vector potentials $\omega_i$ and $A_i$ from the imaginary part of the Ernst scalar potentials $\mathcal{E}$ and $\psi$)!

– Black hole unicity theorems $\implies$ this must be singular!
• The full solution \((x > 1, \ -1 \leq y \leq 1)\)

\[
ds^2 = -\frac{f}{\Sigma} \left( dt - \frac{\kappa \Pi}{f} d\varphi \right)^2 + \kappa^2 \Sigma \left[ e^{2\nu} \left( \frac{dx^2}{x^2-1} + \frac{dy^2}{1-y^2} \right) + f^{-1}(x^2-1)(1-y^2) d\varphi^2 \right],
\]

\[
f = \frac{p^2(x^2-1)^2}{4x^2} - \frac{q^2x^2(1-y^2)}{x^2-1},
\]

\[
\Sigma = \left[ \frac{px^2 + 2x + p}{2x} + \frac{q^2(1-y^2)}{2(x^2-1)} \right]^2 + q^2 \left( \frac{p}{2x} - 1 \right)^2 y^2,
\]

\[
e^{2\nu} = \frac{4x^2(x^2-1)^2}{p^2(x^2-y^2)^3},
\]

\[
\Pi = \Pi_1(x)(1-y^2) + \Pi_2(x)(1-y^2)^2,
\]

\[
A = \frac{\varepsilon}{\Sigma} \left[ \bar{v} dt + \kappa \Theta d\varphi \right],
\]

\[
\bar{v} = v_0(x) + v_1(x)(1-y^2) + v_2(x)(1-y^2)^2,
\]

\[
\Theta = \Theta_1(x)(1-y^2)+\Theta_2(x)(1-y^2)^2+\Theta_3(x)(1-y^2)^3.
\]
• Asymptotically flat metric.

  – mass \( M = 2\kappa/p \),
  
  – angular momentum \( J = -\kappa^2 q(4 + p^2)/p^2 \),
  
  – dipole magnetic moment \( \mu = \varepsilon\kappa^2 q \),
  
  – quadrupole electric moment \( Q_2 = \varepsilon\kappa^3 q^2/p \).

• Possible singularities:

  – ring singularities \( (\rho = \rho_0) \)
    \[
    \Sigma(x, y) = 0 \quad (2 \text{ eqs.})
    \]
    no solution! \( (\Sigma > (1 + p)^2) \)

  – axial singularities \( (\rho = 0) \):
    \[
    \begin{cases}
    \text{segment } R & (x = 1, \ -1 < y < 1), \\
    \text{points} & (x = 1, \ y = \pm 1).
    \end{cases}
    \]
• Ergosurface

\[ F \equiv f / \Sigma = 0. \]

2 components:
\( a) \ f(x, y) = 0, \) contains \( R \ (f < 0); \)
\( b) \ R \) itself \( (\Sigma = \infty). \)

• Causal boundary

\[ g_{\varphi \varphi} \equiv F^{-1} \rho^2 - F\omega^2 = 0 \]
contains \( R \) \( (g_{\varphi \varphi} < 0). \)

• Horizons

\[ N^2 \equiv \rho^2 / g_{\varphi \varphi} = 0 \ (\text{with} \ g_{\varphi \varphi} > 0) \]
\[ \rightarrow \text{candidates} \ H_{\pm}(x = 1, \ y = \pm 1). \]
• **Geodesics:**

1st integral \( T + U = \epsilon \) \((\epsilon = -1, 0, +1)\)

\[
T = \kappa^2 \sum e^{2\nu} \left( \frac{\dot{x}^2}{x^2 - 1} + \frac{\dot{y}^2}{1 - y^2} \right) > 0,
\]

\[
U = \frac{(l - E\omega)^2 F}{\rho^2} - \frac{E^2}{F}.
\]

• **Near** \( R \) \((x = 1, y^2 < 1)\):

- \(-F \propto \xi^2, \ \rho^2 \propto \xi^2\) \((\xi^2 \equiv x^2 - 1 \to 0)\)
- \(E \neq 0\): \(U \gg \epsilon \implies\) geodesics turn back before reaching \( R \).
- \(E = 0\): geodesics terminate on \( R \), but timelike or null geodesics \((\epsilon = -1 \) or \(0)\) cannot originate from \(\infty\):
  “harmless” naked singularity.

• **Near** \( H_{\pm} \): Geodesics such that, near \( x = 1, 1 - y^2 \sim X^2(x^2 - 1)\) \((X\ fixed)\)
can be continued through \(x = \pm y = 1\) to a region with \(x < 1\) and \(y^2 > 1\)
\(\implies\) 2 double horizons.

2 black holes \((H_{\pm})\) held apart by a rod \((R)\).
• **Interlude: 2-black hole stationary solutions**

• **BPS superpositions**
  
  Majumdar, Papapetrou (1947): Static linear superpositions of $N$ identical extremal BH.
  
  Israel + Wilson, Perjès (1971): Stationary linear superpositions of $N$ BH, with rod singularities (Hartle + Hawking, Bonnor + Ward (1972)).

• **Weyl superpositions**
  
  Stationary axisym. linear superpositions of $N$ identical non-extremal BH, with singular rods (Bach + Weyl 1922, Israel + Khan 1964).

• **Extremal diholes**
  
  – Bonnor (1966): Static solution, with only mass and magnetic dipole moment.
  
  – Emparan (2000): This is a dihole: 2 extreme magnetic RN BH, with equal masses and opposite magnetic charges, held apart by a rod (can be replaced by an external magnetic field, at the expense of asymptotic flatness).
• **Double Kerr**  
  – Kramer+Neugebauer 1980: Double Kerr-NUT.  
  – Bonnor+Steadman 2003: For equal masses, this is as flat if either the 2 spins are opposite, or the 2 spins are equal, and the rod between the holes is spinning.  
  – Co-rotating double Kerr with massless, non-spinning rod (only conical singularity): Cabrera-Munguia et al., Manko+Ruiz (2017)

• **Non-extremal diholes**  
  – Emparan+Teo 2001: Static non-extremal diholes with equal masses, opposite charges, and rod.  
  – Generalization to 2 counter-rotating black dyons with opposite charges: Cabrera-Munguia et al. 2013, Manko et al. 2014
• The horizons: geometry

Blow up the horizons $x = \pm y = 1$ by transforming to the coords. (Kodama+Hikida):

$$X = \sqrt{\frac{1 - y^2}{x^2 - 1}}, \quad Y = \frac{y}{x}.$$ 

On the horizons $Y = \pm 1$,

$$F_H = -\frac{q^2 X^2}{\Sigma_H(X)}, \quad \omega_H^{-1} = \Omega_H = -\frac{q}{\kappa \lambda(p)},$$

$$\Sigma_H(X) = \frac{p \lambda(p)}{2} + q^2 (1 + p) X^2 + \frac{q^4}{4} X^4.$$
– In the co-rotating near-horizon frame
\( \hat{\varphi} = \varphi - \Omega_H t \),

\[
\begin{align*}
\frac{d s_H^2}{\kappa^2 \lambda(p)} &= \frac{\lambda(p)}{2 pl(\theta)} \left[ d\theta^2 + l^2(\theta) \sin^2 \theta d\hat{\varphi}^2 \right], \\
l(\theta) &\equiv \frac{p \lambda(p) (X^2 + 1)^2}{2 \Sigma_H(X)}, \quad X = \tan(\theta/2).
\end{align*}
\]

Topologically \( S^2 \).

\( l(0) = 1 \), but \( l(\pi) = \alpha \equiv \frac{2 p \lambda(p)}{q^4} > \frac{8}{q^4} \)

Conical singularity!

– Horizon area:

\[
A_H = 4\pi \frac{\kappa^2 \lambda(p)}{2p} \simeq 4\pi M^2.
\]
• The horizons: electromagnetic field

\[ A_H = \varepsilon \left[ \frac{q^2(2 - p)}{2\lambda(p)} dt - \frac{\kappa q \delta(p) X^2 + q^2 \gamma(p) X^4}{4 \sum_H(X)} d\phi \right] \]

- Near-horizon electric field, or

\[ Q_H = -\frac{1}{4\pi} \oint_H \omega_H d\text{Im}\psi \] (Tomimatsu 1984)

Electric charges \( Q_+ = Q_- = -\frac{\varepsilon \kappa (1 + p)}{2} \).

But the solution is electrically neutral, so the rod must be also charged!

\[ \rightarrow \text{electric quadrupole} \]

- Magnetic charges: \( P_H = \frac{1}{4\pi} \oint_H dA_\phi \)

\[ P_{\pm} = \pm \frac{\varepsilon \kappa \gamma(p)}{2q}. \]

\[ \rightarrow \text{magnetic dipole} \]
- Komar mass and angular momentum:

\[
M = \frac{1}{4\pi} \int k^\mu;\nu d\Sigma_{\mu\nu} \quad (k = \partial_t)
\]

\[
J = -\frac{1}{8\pi} \int l^\mu;\nu d\Sigma_{\mu\nu} \quad (l = \partial_\phi)
\]

- Ostrogradsky theorem \[\rightarrow\]

\[
M = \sum_n \frac{1}{4\pi} \int_{H_n} k^\mu;\nu d\Sigma_{\mu\nu} + \frac{1}{4\pi} \int k^\mu;\nu;\nu dS_\mu + \frac{1}{4\pi} \int R^\mu_\nu k^\nu dS_\mu
\]

- Tomimatsu: using the EM eqs., the bulk integral can be converted to a surface integral

\[\rightarrow\] total horizon mass \( M_H = \frac{1}{4\pi} \int_H \omega_H d\text{Im}\mathcal{E}: \]

\[
M_+ = M_- = \frac{\kappa}{p} + \frac{\kappa p}{2}
\]

\( > M/2 \), so the rod must have negative mass!

- Horizon angular momentum:

\[
J_+ = J_- = -\frac{\kappa^2}{8qp} \left[ 2\lambda(2 + p^2) - q^2 p(1 + p)(2 - p) \right]
\]

- Also horizon NUT charges: \( N_\pm = \pm \frac{\kappa \lambda(p)}{4q} \).
The rod

Near-rod configuration \( (\xi^2 \equiv x^2 - 1 \to 0) \):

\[
\begin{align*}
    ds^2 & \sim -\frac{\kappa^2 q^2}{4} (1 - y^2)^2 d\varphi^2 + \frac{\kappa^2 q^4}{p^2(1 - y^2)} \left[ \frac{dy^2}{1 - y^2} + d\xi^2 + \alpha^2 \xi (dt - \omega(y) d\varphi)^2 \right], \\
    A & \sim -\varepsilon \left[ \left( 1 - \frac{2(1 + p)\xi^2}{q^2(1 - y^2)} \right) dt + A_\varphi(y) d\varphi \right], \\
\end{align*}
\]

\( \to \) conical singularity, with finite Ricci square scalar

\[
R^{\mu\nu} R_{\mu\nu} \sim \frac{64 p^4}{\kappa^4 q^{12}} [(1 + p)^2 + q^2 y^2]^2.
\]

Transformation to the horizon co-rotating frame:

\[
\begin{align*}
    ds^2 & \sim q^4 \left[ -\frac{(1 - y^2)^2}{4 \lambda^2(p)} \left( dt + \Omega_{H}^{-1} d\varphi \right)^2 + \frac{\kappa^2}{p^2(1 - y^2)} \left( \frac{dy^2}{1 - y^2} + d\xi^2 + \alpha^2 \xi^2 d\varphi^2 \right) \right].
\end{align*}
\]
• Interlude: Straight spinning cosmic string in flat spacetime:

(Deser, Jackiw, ’t Hooft 1984, GC 1985)

\[ ds^2 = -(dt + 4JS \, d\hat{\phi})^2 + d\rho^2 + \alpha^2 \rho^2 \, d\hat{\phi}^2 + dz^2 \]

(\( \alpha = 1 - 4MS \)).

The same viewed in a rotating frame,

\[ d\phi = d\hat{\phi} - \Omega \, dt \]

with critical angular velocity \( \Omega = -1/4JS \):

\[ ds^2 = \alpha^2\Omega^2\rho^2(dt+\Omega^{-1} \, d\phi)^2+d\rho^2-\Omega^{-2}d\phi^2+dz^2. \]
– The rod is a spinning cosmic string in curved spacetime, with negative tension ($\alpha > 1$).

– This “spinning” string is a Misner string connecting 2 opposite NUT sources at $z = \pm \kappa$. We do not periodically identify time (Clément, Gal’tsov, Guenouche, “Rehabilitating space-times with NUTs”, Phys. Lett. B750 (2015) 591, arXiv:1508.07622)

1 NUT source: $\omega = -2N \cos \theta$

2 opposite NUT sources

$\omega = -2N \cos \theta_+ + 2N \cos \theta_-$

On the axis ($\rho = 0$):

$\omega = -4N (-\kappa < z < \kappa)$ or $0 (|z| > \kappa)$ \Rightarrow

$$N = J_S = -\frac{1}{4\Omega_H} = \frac{\kappa \lambda(p)}{4q}$$
• Rod vector potential

\[ A_\phi \sim -\varepsilon \kappa \left[ \frac{\gamma(p)}{q} + \frac{q(1 - y^2)}{2} \right] \]

– Constant contribution \(-\varepsilon \kappa \frac{\gamma(p)}{q} = P_- - P_+\) :

The rod is also a Dirac string connecting 2 opposite magnetic monopoles at \(z = \pm \kappa\)

– Radial magnetic flux density

\[ \sqrt{|g|} B_\xi = F_{y\phi} = \varepsilon \kappa q y \]

\[ \rightarrow \text{rod magnetic moment} \]

\[ \mu_R = \frac{1}{4\pi} \int_{-1}^{+1} \sqrt{|g|} B_\xi z 2\pi dy = \frac{\varepsilon \kappa^2 q}{3} = \frac{\mu}{3} \]
• Rod electric charge

The Maxwell equation \( \partial_\nu(\sqrt{|g|}F^{\mu\nu}) = 0 \) is satisfied only outside sources \( \rightarrow \) distributional contribution

\[
Q_R = \frac{1}{4\pi} \int \left[ \partial_\xi \left( \sqrt{|g|}F_{t\xi} \right) \right] d\xi \, dy \, d\phi.
\]

In the global frame,

\[
A_t = -\varepsilon[1 + O(\xi^2)], \quad g_{tt} = O(\xi^2) \quad \rightarrow
\]

\[
F_{t\xi} \propto \xi \quad (\xi > 0), \quad \sqrt{|g|}F_{t\xi} \propto \theta(\xi)
\]

\[
\Rightarrow Q_R = \frac{1}{4\pi} \int \varepsilon \kappa(1+p)\delta(\xi) \, d\xi \, dy \, d\phi = \varepsilon \kappa(1 + p)
\]

ensures \( Q_+ + Q_- + Q_R = 0. \)
• Rod mass

The Einstein eqs. with Maxwell source $R_{\mu\nu} - 8\pi T_{\mu\nu} = 0$ are only satisfied outside sources. In the presence of distributional sources,

$$R_{\mu\nu} - 8\pi T_{\mu\nu} = [R_{\mu\nu}] - 8\pi [T_{\mu\nu}]$$

⇒ Komar mass at $\infty$

$$M = M_+ + M_- + M_R \quad \text{with}$$

$$M_R = -\frac{1}{4\pi} \int \left( [R^t_t] - 8\pi [T^t_t] \right) \sqrt{|g|} d^3x = M_{R}^{\text{grav}} + M_{R}^{\text{em}}$$

$$[R^t_t] = -(g_{tt})^{-1/2} g^{ij} (g_{tt})^{1/2} = -g^{\xi\xi} \delta(\xi)$$

→ $M_{R}^{\text{grav}} = \kappa$

$M_{R}^{\text{em}} = Q_R A_t (\xi = 0) = -\kappa (1 + p)$

− Total rod mass $M_R = -\kappa p$

repulses test particles (antigravity)

$$\frac{2\kappa}{p} = 2 \left( \frac{\kappa}{p} + \frac{\kappa p}{2} \right) - \kappa p$$
• **Behind the horizon** $H_+$:
  - Region $II_+$ with $-1 < x < 1$ and $y > 1$
  - Inner horizon $H'_+$ ($x = -1, y = 1$)

• Between outer and inner horizon, timelike singularity $S_0$ ($y = \infty$), with $f < 0$, $g_{\varphi\varphi} < 0$, and Ricci square scalar $\sim y^4$

• Is $S_0$ really at infinity?
  - The 2 horizons $H$ and $H'$ are topological spheres, with $A > A'$
  - Near $S_0$, putting $y = \eta^{-1}, x = \cos \chi$,
    $$ds^2 \simeq -\frac{a}{\cos^2 \chi} \eta^{-4} d\varphi^2 + b \cos^2 \chi [d\eta^2 + \eta^2 d\chi^2$$
    $$+ c \eta^2 \sin^2 \chi (dt + k(x) \eta^{-2} d\varphi^2)]$$
    $\implies \sqrt{|g|}$ goes to a finite limit for $y \to \infty$
    - $\eta = 0$ is a ”point” (timelike line)

• Only spacelike geos with $E = 0$ terminate at $y = \infty$. 
• Now the $z$ axis ($\rho = 0$) includes:
  – a regular segment $y = 1, -1 < x < 1$ between the 2 horizons;
  – 2 singular rods $x = \pm 1$ from the outer or inner horizon to $S_0$.
  – The 2 rods have different tensions and angular velocities.
  – The 2 rods carry different diverging electric charges (integration on $y$ from 1 to $\infty$), so $S_0$ must carry infinite electric charge, and finite magnetic and NUT charges.

• Similar region $II_-$ ($y < -1$) behind $H_-$.

• **Beyond the inner horizons:**
  – Region $III$ ($X < -1, -1 \leq y \leq 1$), with a singular rod connecting the 2 co-rotating horizons $H'_\pm$ and $H'_-$, and
  – A timelike ring singularity $\Sigma(x_0, 0) = 0$, with $f > 0, g_{\varphi\varphi} < 0$.
  – Only fine-tuned spacelike geodesics can reach this ring (similar to Kerr-Newman).
• **Summary**

– Exact one-parameter rotating e.m. solution generated from ZV2 static vacuum solution.
– No naked ring singularity: more regular than the seed static solution, or its rotating vacuum counterpart TS2.
– Has only dipole magnetic moment and quadrupole electric moment.
– Generated by a complex system: 2 co-rotating dyonic NUTty black holes held apart by a rotating, electrically charged, magnetized rod.
– More general 4-parameter class of solutions (Manko et al 2000): Does it include a purely magnetic rotating solution (without quadrupole electric moment) → a system of 2 magnetic NUTty black holes and an electrically neutral rod?