Weyl-Orborn Anomalies and Soliton masses

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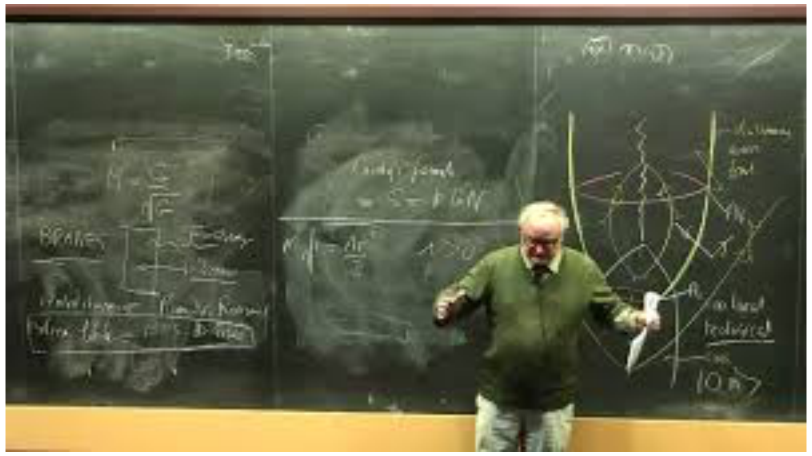
GARYFEST



Le Studium Conference, Tours, March 22-24

I feel very lucky, and honored, to be here for this celebration of Gary's brilliant career and achievements

Gary is the closest approximation,

within our community,

to a Renaissance man

Knows (almost) everything and interested in everything

SCHOLARPEDIA

before its time !

Linear A

Brunalleschi's
self-supporting dome

. . . .

his Schwarzschild radius

Gary's papers: word occurence in titles

black hole 65
metric 36 (geometry 23)
branes 35
quantum 31
(anti) de Sitter 18
solitons 17
string 17
gravity 15
horizon 9
moduli 6
Euclidean 6
Born-Infeld 6
Carroll 4

Gibbons-Manton metric for N monopoles

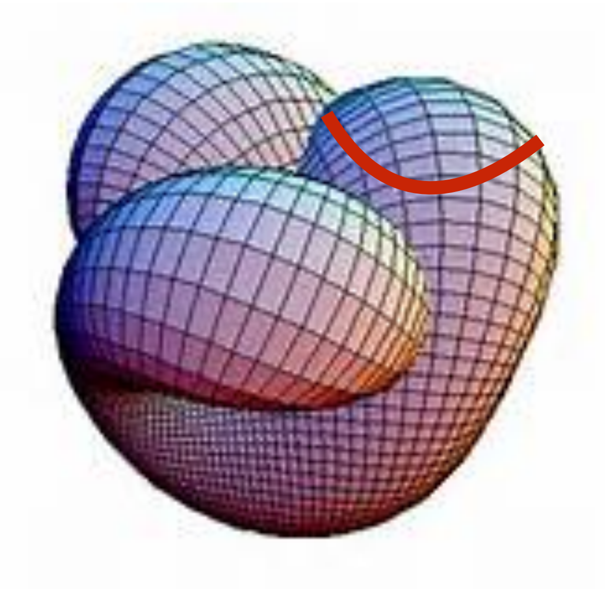
Attractor-mechanism for BHs (flux compacts)

1. Introduction

Today's talk will be about a famous moduli space:

that of **Calabi-Yau manifolds**

and its associated solitons: **wrapped D-branes**



$\times \quad \mathbb{R}^{1,3} \quad \simeq$ **Our world ?**

Moduli (λ) : massless scalar fields; metric determines eff. action

D-branes: solitons with λ - dependent mass and charge

Computing $g_{I\bar{J}}(\lambda)$ and $M_s(\lambda)$ is time-honored problem in physics and mathematics.

Gromov-Witten invariants

Recent progress in **susy field theories** has opened a new avenue for the solution of these problems.

They are related to **2D Weyl anomalies**, which shows in turn that they can be computed by the technique of **localization**

Pestun 2007

. . . .

Benini, Cremonesi 1206.2356

Doroud, Gomis, Le Floch, Lee 1206.2606

Based on dictionary:

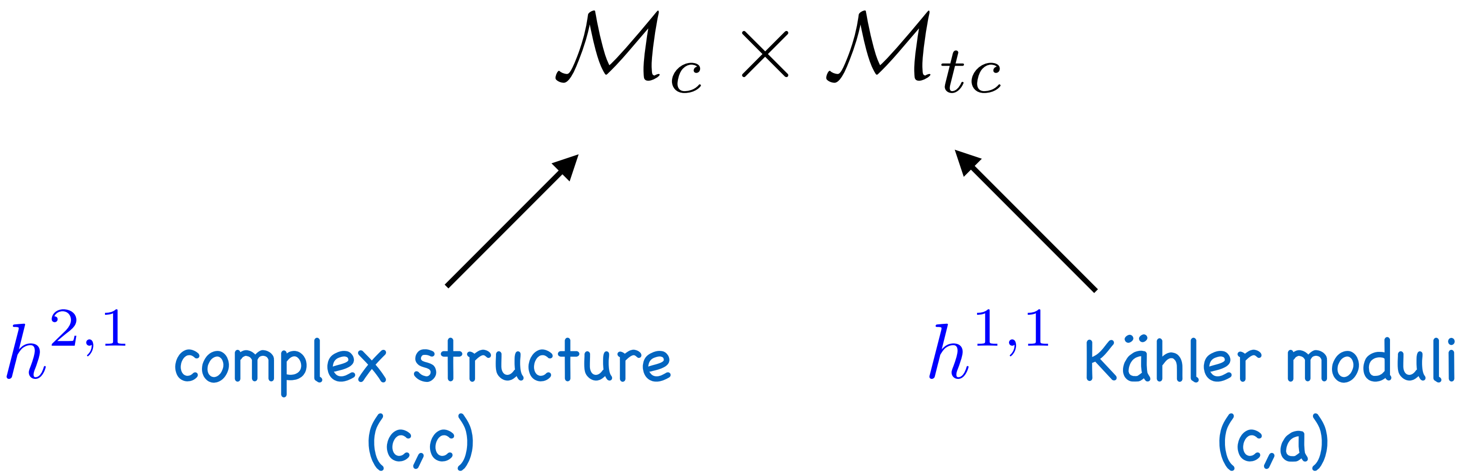
Target space	**worldsheet**
Calabi-Yau	N=(2,2) SCFT
moduli λ^I	marginal deformations
moduli space	superconformal manifold
metric	Zamolodchikov metric
wrapped brane	boundary conditions Ω
mass	bnry degeneracy g^Ω
	Affleck, Ludwig

New way to compute r.h.s.

More precisely :

This CY moduli space is known to **factorize locally**:

but see arXiv:1611.03101
Gomis, Komargodski, Ooguri, Seiberg, Wang



$$\mathcal{M}_c \times \mathcal{M}_{tc}$$

$h^{2,1}$ complex structure
(c,c)

$h^{1,1}$ Kähler moduli
(c,a)

What is the metric on this moduli space ?

Strong constraints from $\mathcal{N} = 2$ supersymmetry of type-II string theory compactified on CY3:

$$\begin{array}{llll} IIA: & h^{1,1} \ \ vector & h^{2,1}+1 & hyper \\ IIB: & h^{2,1} \ \ vector & h^{1,1}+1 & hyper \end{array}$$

special Kähler quaternionic

The string coupling is a hyper, while the volume is a Kähler modulus

$\Longrightarrow$

metric on complex-structure m.s. is **classical**

metric on Kähler m.s. has **instanton** corrections

Gromov-Witten invariants

It has been shown that the Kähler m.s. metric is

computed by the **partition function on the 2-sphere**

$$\boxed{Z(S^2) = \left(\frac{r}{r_0}\right)^{c/3} e^{-K(\lambda,\bar{\lambda})}}$$

$$g_{I\bar{J}} = \partial_I \partial_{\bar{J}} K$$

Conjectured by Jockers, Kumar, Lapan, Morrison, Romo (1208.6244)

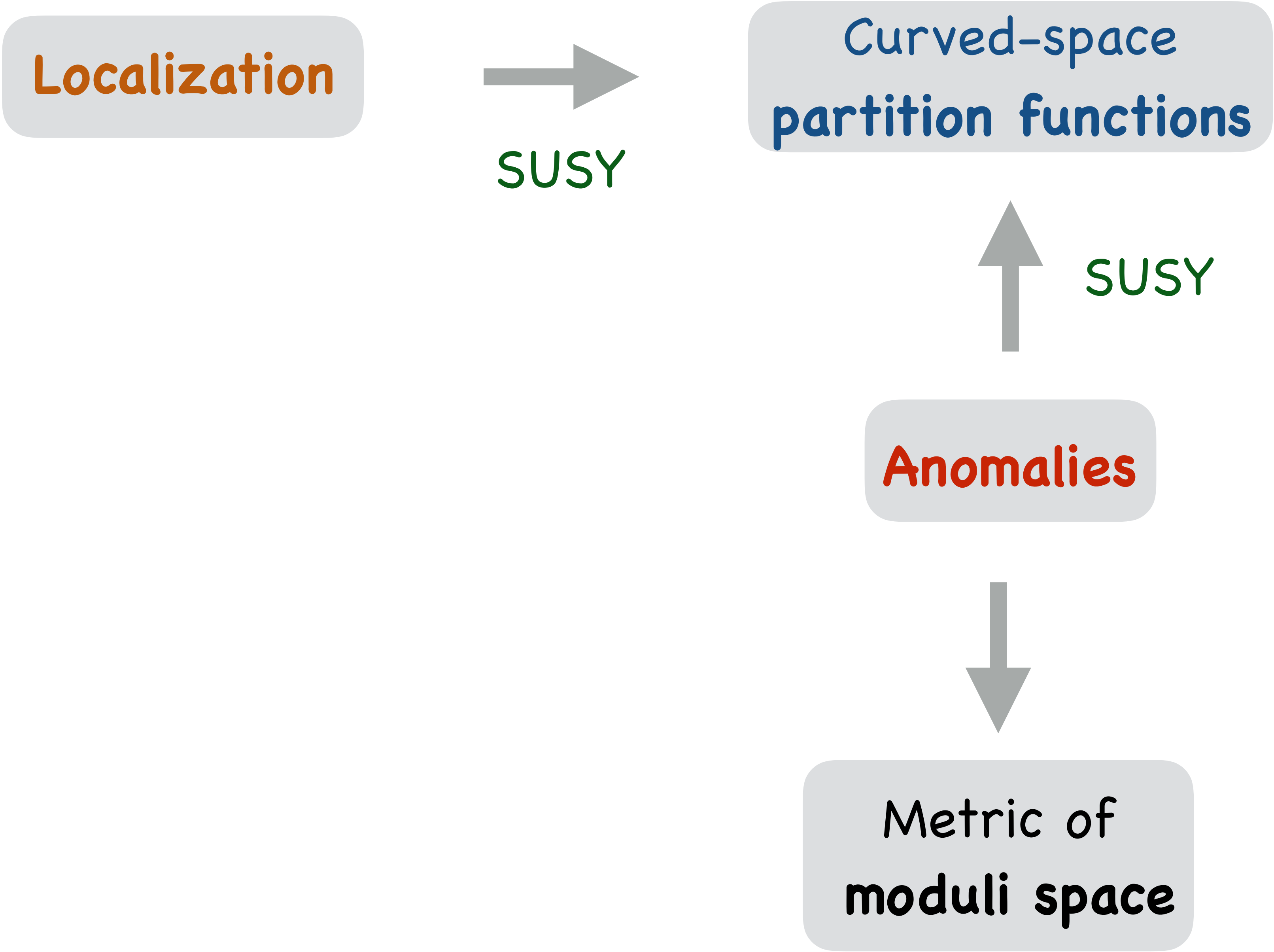
Shown with help of tt* eqns by Gomis + Lee (1210.6022)

Last year: Gomis, Komargodski, Hsin, Schwimmer, Seiberg, Theisen (1509.08511)

gave an elegant new proof based on a ``new kind" of **Weyl anomaly**

Osborn '91

This is a powerful circle of ideas with non-trivial corollaries and generalizations to higher D:



Localization

SUSY

Curved-space **partition functions**

SUSY

Anomalies

Metric of **moduli space**

With **Daniel Plencner** we generalized this line of argument to the **hemisphere**

arXiv: 1612.06386

We show in particular that the hemishpere partition function computes the second important piece of geometric data:

The **central charge** $C^{\Omega}(\lambda)$, and the **mass** of CY D-branes.

Honda + Okuda 1308.2217
Hori + Romo 1308.2438
Sugishita + Terashima 1308.1973

Rest of this talk:

② Superconformal manifolds & Anomalies

③ Corollaries for $\mathcal{Z}(S^2)$ and CY moduli space

④ Extension to Boundaries and D-brane mass/charge

⑤ Summary

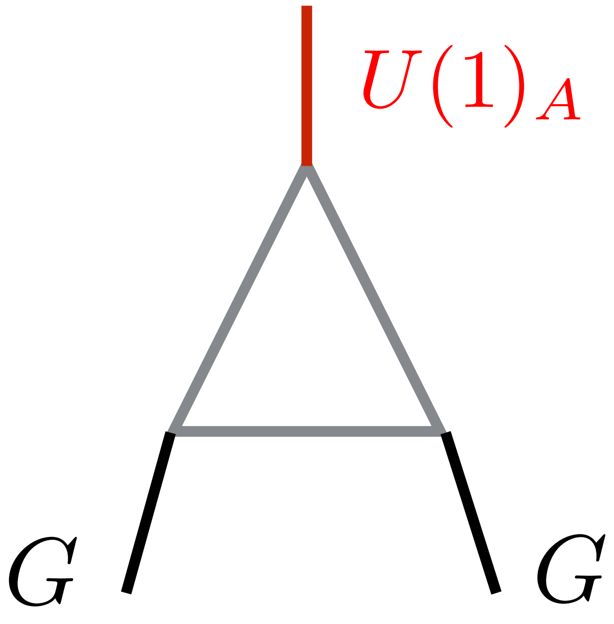
2. ``Mixed" super-Weyl anomalies

Anomalies imply non-conservation in correlation functions:

$$\langle \partial_\mu j^\mu \, \mathcal{O}_1(p_1) \cdots \mathcal{O}_n(p_n) \rangle \neq 0$$

When r.h.s. proportional to momenta: non-conservation only in presence of **spacetime-dependent background fields**

e.g.



$$\Longrightarrow \quad \partial_\mu j^\mu_A = F \wedge F$$

axial charge violated by instantons, cf 't Hooft matching conditions

For chiral anomalies: background is gauge or gravitational

For trace (Weyl) anomaly, can be **exactly-marginal couplings**:

Osborn '91
Osborn, Petkou '93

In 2D the 2-point function of marginal operators reads:

Zamolodchikov metric

anomaly

$$\langle \mathcal{O}_I(z)\bar{\mathcal{O}}_{\bar J}(w)\rangle = g_{I\bar J}\,\mathcal{R}\frac{1}{|z-w|^4} = g_{I\bar J}\,\frac{1}{2}(\partial\bar\partial)^2\left[\log(|z-w|^2\mu^2)\right]^2$$

differential regularization of distribution

Turn on space-dependent couplings λ^I :

$$\frac{\partial \mathcal{Z}}{\partial \log \mu} \sim \int_z \int_w \lambda^I(z) \bar{\lambda}^{\bar{J}}(w) \frac{\partial}{\partial \log \mu} \langle \mathcal{O}_I(z) \bar{\mathcal{O}}_{\bar{J}}(w) \rangle \sim \int \partial_\mu \lambda^I \partial^\mu \bar{\lambda}^{\bar{J}} g_{I\bar{J}}$$

Anomaly is **invisible for constant couplings**. But supersymmetry relates it to a term that does not vanish when $\partial_\mu \lambda^I = 0$

Gomis et al (1509.08511)

This term can be removed by non-susy local counterterm;

but **SUSY gives it universal meaning**

Technical details:

To $\mathcal{N}=(2,2)$ SCFTs have $U(1)_V \times U(1)_A$ R-symmetry.

In computing the anomaly we choose to preserve the **vector-like** symmetry, so we must couple it to the $\mathcal{N}=2$ supergravity in which this symmetry is gauged by a field V^μ

Closset + Cremonesi (1404.2636)

In superconformal gauge:

$$g_{\mu\nu} = e^{2\sigma}\eta_{\mu\nu} \; , \qquad V^\mu = \epsilon^{\mu\nu}\partial_\nu a$$

Classically σ and a decouple, but in the quantum theory they dont due to the **Weyl** and **axial** anomalies.

Supersymmetry places these fields in a twisted-chiral multiplet

$$\Sigma(y^\mu) = (\sigma + ia) + \theta^+ \bar{\chi}_+ + \bar{\theta}^- \chi_- + \theta^+ \bar{\theta}^- w$$

with components functions of $y^\pm = x^\pm \mp i\theta^\pm \bar{\theta}^\pm$

The tc field obeys $\overline{D}_+ \Sigma = D_- \Sigma = 0$

where $$D_\pm = \frac{\partial}{\partial \theta^\pm} - i\bar{\theta}^\pm \partial_\pm \ , \qquad \overline{D}_\pm = -\frac{\partial}{\partial \bar{\theta}^\pm} + i\theta^\pm \partial_\pm \ .$$

It is useful to also promote the marginal couplings to vevs of tc fields

$$\Lambda^I = \lambda^I(y^\pm) + \cdots \ , \qquad \bar{\Lambda}^I = \bar{\lambda}^I(\bar{y}^\pm) + \cdots$$

Seiberg

so as to make the susy of the anomaly manifest.

The anomaly $iA(\delta\Sigma) := \delta_\Sigma \log \mathcal{Z}_V(M)$ is the susy invariant

$$A_{\text{closed}} := A^{(1)} + A^{(2)} = \frac{1}{4\pi}\int_M d^2x \int d^4\theta \left[\frac{c}{6}(\delta\Sigma\,\bar{\Sigma} + \delta\bar{\Sigma}\,\Sigma) - (\delta\Sigma + \delta\bar{\Sigma})K(\Lambda,\bar{\Lambda})\right]$$

Gomis et al (1509.08511)

This obeys **Wess-Zumino** consistency $\delta_\Sigma A(\delta\Sigma') - \delta_{\Sigma'} A(\delta\Sigma) = 0$

and can be integrated with the result:

$$\log \mathcal{Z}_V \supset \frac{i}{4\pi}\int_M d^2x \int d^4\theta \left[\frac{c}{6}\Sigma\,\bar{\Sigma} - (\Sigma + \bar{\Sigma})K\right] .$$

super-Liouville

super-**Osborn** '91

Expand in components:

$$A^{(1)} = -\frac{c}{12\pi}\int_M d^2x \left[\delta\sigma\,\Box\sigma + \delta a\,\Box a + \frac{1}{2}(\delta w\,\bar{w} + \delta\bar{w}\,w) + \partial^\mu b^{(1)}_\mu\right] \; + \text{fermions}\;,$$

$$A^{(2)} = -\frac{1}{2\pi}\int_M d^2x \left[\delta\sigma\,(\partial_\mu\lambda^I\partial^\mu\bar{\lambda}^{\bar{J}})\partial_I\partial_{\bar{J}}K - \frac{1}{2}K\,\Box\delta\sigma - (\partial^\mu\delta a)\mathcal{K}_\mu + \partial^\mu b^{(2)}_\mu\right]$$

where $\mathcal{K}_\mu := \frac{i}{2}(\partial_I K\partial_\mu\lambda^I - \partial_{\bar{I}}K\partial_\mu\bar{\lambda}^{\bar{I}})$ ← Kahler one-form

🟦 (Cohomologically) **non-trivial**, real anomalies

🟧 Variation of **local invariant counterterm**

$$\sim \int \sqrt{g}R^{(2)}\,K(\lambda,\bar{\lambda})$$

The first term in $A^{(2)}$ is the **scale anomaly in the 2-point function**

as follows from $\delta\sigma = -\delta \log \mu \ , \quad \partial\bar{\partial} \log |z|^2 = \pi \delta^{(2)}(z)$

and $\partial_I \partial_{\bar{J}} K = g_{I\bar{J}}$

contact term

The non-vanishing term for constant couplings is the **red** one

It could be removed by **change of scheme** in bosonic theory,

but supersymmetry relates it to the non-trivial **blue** terms !

Similar remark made previously for 4D Casimir energy by

Assel, Cassani, Di Pietro, Komargodski, Lorenzen, Martelli 1503.05537

3. Some corollaries

Integrating the anomaly for constant couplings gives

$$\int_{S^2} K\,\Box\sigma = -4\pi K \quad \Longrightarrow \quad \boxed{Z_V^E(S^2) = \left(\frac{r}{r_0}\right)^{c/3} e^{-K(\lambda,\bar{\lambda})}}$$

so the 2-sphere free energy computes the Kähler potential

on the SCFT2 moduli space (both chiral and twisted chiral)

A puzzle

$Z_V^E(S^2)$ not invariant under **Kähler-Weyl** transformations

$$K'(\lambda,\bar{\lambda}) \;=\; K(\lambda,\bar{\lambda}) + H(\lambda) + \bar{H}(\bar{\lambda})$$

Resolution

The difference amounts to change of **renormalization scheme**:

$$\Delta_{\mathrm{KW}} A^{(2)} = -\frac{1}{4\pi}\int_M d^2x \int d^4\theta \, (\delta\Sigma + \delta\bar{\Sigma}) H + c.c.$$

$$= \; -\frac{1}{4\pi}\int_M d^2x \int d\theta^+ d\bar{\theta}^- \, (\bar{D}_+ D_- \delta\bar{\Sigma}) H + \int_M d^2x \; (\partial^\mu Y_\mu) + c.c.$$

twisted F-term

$$\mathcal{R} = \bar{D}_+ D_- \bar{\Sigma} = \; -\bar{w} \; + \; \theta^+\bar{\theta}^- \partial_+ \partial_- (\sigma - ia) + \cdots$$

curvature superfield

So **local, susy and diffeo-invariant counterterm** compensates
the Kähler-Weyl (gauge) transformation !

An interesting conjecture

Gomis et al (1509.08511)

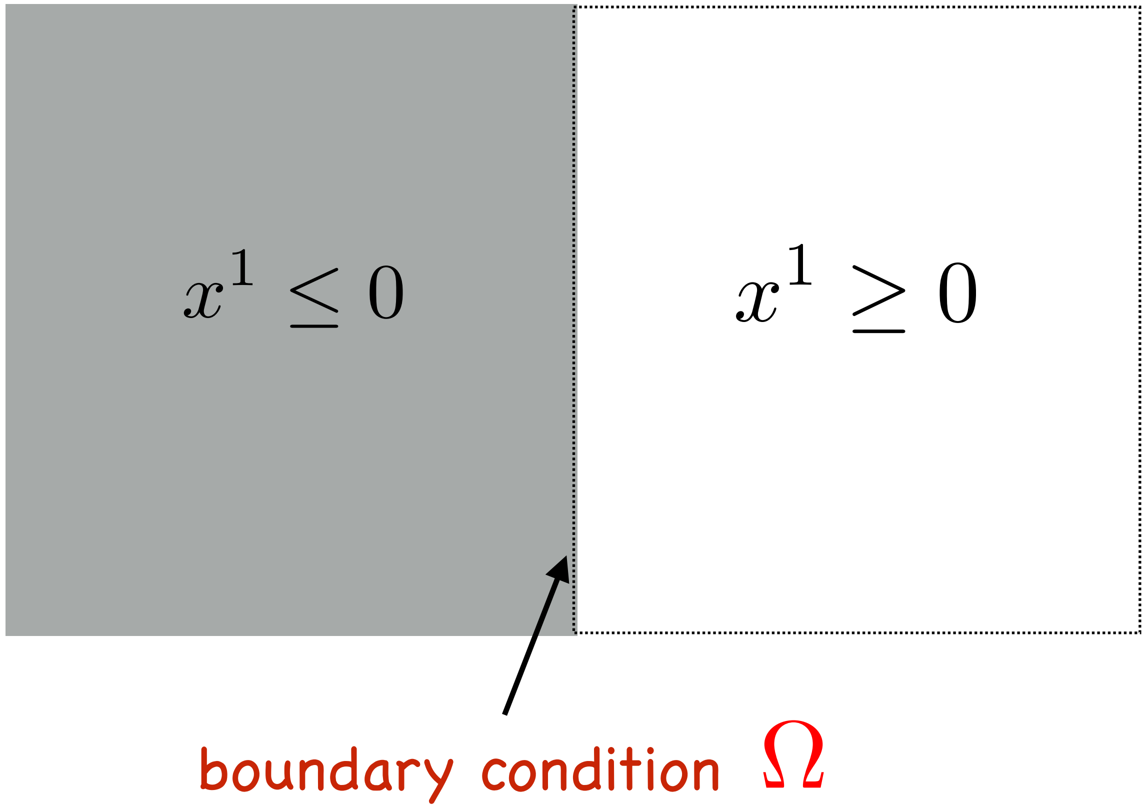
If the moduli space had non-vanishing Kähler class one could pick $\lambda^I(x)$ such that $S^2 \to \mathcal{M}$ is non-trivial 2-cycle

Then there would be no global renormalization scheme !

Way out: **Moduli space has Kähler class = 0**

4. Boundary anomaly

Consider half space:



One-point functions of marginal operators:

$$\langle \mathcal{O}_I(x) \rangle_\Omega = d_I^\Omega \mathcal{R} \frac{1}{|x_1|^2} = d_I^\Omega \, \partial_1^2 \, [\Theta(-x^1) \log |x^1 \mu| \,] \Omega$$

For susy-preserving (B-type) boundary conditions, the 1pt-function coefficients are not independent. They are related to a **holomorphic boundary charge** $c^{\Omega}(\lambda)$

$$\boxed{\frac{c_I^{\Omega}}{c^{\Omega}} = \partial_I (K + \log c^{\Omega})}$$

Ooguri, Oz, Yin '96

<u>Argument</u>: vacuum projection of boundary state

$$\Pi_{\text{vac}}\, |\Omega\rangle\rangle := c^{\Omega}\, |0\rangle_{\text{RR}} + \sum_I c_I^{\Omega}\, |I\rangle_{\text{RR}}$$

is flat section of the improved connection $\nabla - C$ on moduli space

structure constants of chiral ring

Our result: prove these relations from Weyl-Osborn anomaly, and show that hemisphere p.f. computes bnry charge

$$\mathcal{Z}_+(D^2) = \left(\frac{r}{r_0}\right)^{c/6} c^\Omega(\lambda) \;, \qquad \mathcal{Z}_-(D^2) = \left(\frac{r}{r_0}\right)^{c/6} c^\Omega(\bar{\lambda}) \;.$$

Under Kähler Weyl transformations $c^\Omega \;\to\; c^\Omega \, e^F$

The **boundary entropy** is the scheme-independent combination

$$g^\Omega = \frac{|c^\Omega|}{e^{-K/2}} \;=\; \sqrt{\frac{\mathcal{Z}_+(D^2)\mathcal{Z}_-(D^2)}{\mathcal{Z}(S^2)}}$$

In string-theory compactifications, g^{Ω} and c^{Ω} are the **mass** and **RR charge** of the 1/2 BPS D-brane states

↑

dyons in field-theory limits

These follow from worldsheet anomalies !

Technical details:

3 steps in calculation:

- Take into account the divergence terms in $A_{\rm closed}$

$$b_\mu^{(1)} = \frac{1}{4}(\partial_\mu \delta\sigma)\sigma - \frac{3}{4}\delta\sigma\,\partial_\mu\sigma + \frac{1}{4}(\partial_\mu\delta a)a - \frac{3}{4}\delta a\,\partial_\mu a$$

$$b_\mu^{(2)} = \frac{1}{4}(\partial_\mu\delta\sigma)K - \frac{1}{4}\delta\sigma\,\partial_\mu K \ .$$

- Add `minimal' boundary term needed for susy

- Extra boundary-superinvariant additions using formalism of **boundary superspace**

Reference boundary completion

top component

Consider the D-term :

$$\boxed{\int_M d^2x \int d^4\theta\, \mathcal{S} = \int_M d^2x\, [\mathcal{S}]_{\rm top}}$$

The **type-B susy** generator is $\mathcal{D}_{\rm susy} = \epsilon\,(Q_+ + Q_-) - \bar{\epsilon}\,(\bar{Q}_+ + \bar{Q}_-)$

where $$\mathcal{Q}_\pm = \frac{\partial}{\partial\theta^\pm} + i\bar{\theta}^\pm\partial_\pm\ , \qquad \overline{\mathcal{Q}}_\pm = -\frac{\partial}{\partial\bar{\theta}^\pm} - i\theta^\pm\partial_\pm$$

The transformation of the D-term is a total derivative

$$\Delta_{\rm susy}[\mathcal{S}]_{\rm top} = \int d^4\theta\ \mathcal{D}_{\rm susy}\mathcal{S} = i\epsilon\int d^4\theta\ (\bar{\theta}^+\partial_+\mathcal{S} + \bar{\theta}^-\partial_-\mathcal{S})\ +\ c.c.$$

We want to write as the susy transformation of a boundary term.

Standard manipulations give:

$$\Delta_{\rm susy}[\mathcal{S}]_{\rm top} = -\Delta_{\rm susy}(\partial_1 [S]_{\rm bnry}) + \partial_0 Y$$

with $$[\mathcal{S}]_{\rm bnry} = -\frac{i}{2}\left([\mathcal{S}]_{\theta^+\bar\theta^-} - [\mathcal{S}]_{\theta^-\bar\theta^+}\right) - \frac{1}{4}\partial_1 [\mathcal{S}]_{\emptyset}$$

so that $$\boxed{\mathrm{I}_D(\mathcal{S}) := \int d^2x\,[\mathcal{S}]_{\rm top} + \int dx^0\,[S]_{\rm bnry}}$$

is our susy-invariant standard completion.

For the case of interest, the integrated superfield is $\delta\mathcal{S}$

with $$\mathcal{S} = \frac{1}{4\pi}\left[\frac{c}{6}\Sigma\bar\Sigma - (\Sigma + \bar\Sigma)K\right]$$

→ Boundary superspace

Hori (hep-th/0012179)

$$x^+ = x^- \,, \qquad \theta \equiv e^{-i\beta}\,\theta^+ = e^{i\beta}\,\theta^- , \qquad \bar\theta \equiv e^{i\beta}\,\bar\theta^+ = e^{-i\beta}\,\bar\theta^-$$

Restrictions of bulk superfields, e.g.

$$\Sigma|_{\partial M} = \sigma + ia + \theta\bar\chi_+ + \bar\theta\chi_- + \theta\bar\theta[w - i\partial_1(\sigma + ia)]$$

Usual D-term and F-term integrals of bnry superfields are invariant

Brunner + Hori (hep-th/0303135)

WZ-consistency, locality and parity covariance leads to ansatz for

boundary-superinvariant contribution to anomaly:

$$\int dx^0\,[\mathcal{B}]_{\theta\bar\theta} \qquad \text{where} \qquad \mathcal{B} = \frac{i}{8\pi}\left[\#\frac{c}{12}(\Sigma^2 - \bar\Sigma^2) + \bar\Sigma\,G^\Omega(\Lambda,\bar\Lambda) - \Sigma\,G^\Omega(\bar\Lambda,\Lambda)\right]\Bigg|_{\partial M}$$

and reality condition $\quad G^\Omega(\bar\Lambda,\Lambda) = [G^\Omega(\Lambda,\bar\Lambda)]^\star$

Collecting everything:

$$\boxed{A_{\rm open} = \int_M d^2x\,[\delta\mathcal{S}]_{\rm top} + \int_{\partial M} dx^0\,\left([\delta\mathcal{S}]_{\rm bnry} + [\delta\mathcal{B}]_{\theta\bar\theta}\right)}$$

where
$$\mathcal{S} = \frac{1}{4\pi}\left[\frac{c}{6}\Sigma\bar\Sigma - (\Sigma + \bar\Sigma)K\right]$$

$$[\mathcal{S}]_{\rm bnry} = -\frac{i}{2}\left([\mathcal{S}]_{\theta^+\bar\theta^-} - [\mathcal{S}]_{\theta^-\bar\theta^+}\right) - \frac{1}{4}\partial_1\,[\mathcal{S}]_{\emptyset}$$

$$\mathcal{B} = \frac{i}{8\pi}\left[\#\frac{c}{12}(\Sigma^2 - \bar\Sigma^2) + \bar\Sigma\,G^{\Omega}(\Lambda,\bar\Lambda) - \Sigma\,G^{\Omega}(\bar\Lambda,\Lambda)\right]\Bigg|_{\partial M}$$

🟥 central-charge anomaly

🟩 Weyl-Osborn anomaly

cf Polchinski; Solodukhin for higher D

Susy Ward identity: $\langle \int \delta \mathcal{L}_{\mathrm{sugra}} \int \delta \mathcal{L}_{\mathrm{SCFT}} \rangle = 0$ if $\delta \bar{\Sigma} = \bar{\Lambda}^I = 0$

$\Longrightarrow$ no terms propto $\delta \Sigma \, \Lambda^I$

$$\Longrightarrow \quad G^{\Omega}(\lambda, \bar{\lambda}) = K(\lambda, \bar{\lambda}) + 2 \log c^{\Omega}(\lambda)$$

Kähler-Weyl covariance (up to local counterterms) requires

$c^{\Omega} := e^{h^{\Omega}}$ section of holomorphic line bundle

$$K \to K + H + \bar{H} \qquad\qquad h^{\Omega} \to h^{\Omega} - H$$

final ingredient: susy hemisphere

. . . . Seiberg, Festuccia 1105.0689

$$A_{\text{open}} \supset \delta \left\{ -\frac{1}{4\pi}\int d^2x \left[\Box(\sigma - ia)h^{\Omega} + \Box(\sigma + ia)\bar{h}^{\Omega}\right] + \frac{i}{4\pi}\int dx^0 \left[\bar{w}\, h^{\Omega} - w\, \bar{h}^{\Omega}\right]\right\}$$

integrated anomaly subtracted so as to vanish for infinitesimal disk depends only the holomorphic boundary charge, plus the auxiliary field of the metric.

Killing-spinor equations imply

$$\begin{aligned} w =& \quad 2i\frac{\zeta^-}{\zeta^+}\partial_z(\sigma + ia + \log\zeta^-) = \quad 2i\frac{\bar\zeta^+}{\bar\zeta^-}\partial_{\bar z}(\sigma + ia + \log\bar\zeta^+)\,, \\ \bar w =& - 2i\frac{\zeta^+}{\zeta^-}\partial_{\bar z}(\sigma - ia + \log\zeta^+) = -2i\frac{\bar\zeta^-}{\bar\zeta^+}\partial_z(\sigma - ia + \log\bar\zeta^-)\,. \end{aligned}$$

where the unbroken superconformal symmetries are

$$\epsilon_+ = \epsilon\,\zeta^-(z)\ , \quad \epsilon_- = -\epsilon\,\zeta^+(\bar z)\ , \quad \bar\epsilon_+ = \bar\epsilon\,\bar\zeta^-(z)\ , \quad \bar\epsilon_- = -\bar\epsilon\,\bar\zeta^+(z)$$

Two solutions for hemisphere with B-type bnry condition:

$$\begin{aligned} (+):& \quad \zeta^- = 1, \quad \zeta^+ = \bar z, \qquad \bar\zeta^- = z, \quad \bar\zeta^+ = 1, \\ (-):& \quad \zeta^- = z, \quad \zeta^+ = -1, \quad \bar\zeta^- = 1, \quad \bar\zeta^+ = -\bar z \end{aligned}$$

Supersymmetric hemispheres with B-type bnry condition:

$$\sigma = -\log(1+z\bar{z}) + \text{constant} \ , \qquad a = 0$$

$$(+): \quad w = \bar{w} = -\frac{2i}{1+z\bar{z}} \ , \qquad (-): \quad w = \bar{w} = \frac{2i}{1+z\bar{z}}$$

which implies

$$Z_+(D^2, \Omega) = \mathcal{Z}_0 \, c^\Omega(\lambda) \, , \qquad Z_-(D^2, \Omega) = \mathcal{Z}_0 \, c^\Omega(\bar{\lambda}) \, .$$

qed

5. Summary + outlook

- Computed the super-Weyl anomaly for $\mathcal{N} = (2, 2)$ models on a surface with boundary generalizing the result of

  Gomis, Komargodski, Hsin, Schwimmer, Seiberg, Theisen (1509.08511)

- Not only the **Kähler potential** but also the brane **charge & mass** are given by an (`Osborn-type’) anomaly. They can be computed by localization

- Extension to higher dimensions and other co-dimension defects

Many thanks to the organizers

and congratulations to Gary !