Weyl–Orborn Anomalies and Soliton masses

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GARYFEST

Le Studium Conference, Tours, March 22-24
I feel very lucky, and honored, to be here for this celebration of Gary’s brilliant career and achievements.

Gary is the closest approximation, within our community, to a Renaissance man.
Brunelleschi’s self-supporting dome

Before its time!

Linear A

Knows (almost) everything and interested in everything.

... his Schwarzschild radius
Gary’s papers: word occurrence in titles

black hole 65
metric 36 (geometry 23)

branes 35
quantum 31
(anti) de Sitter 18

solitons 17
string 17
gravity 15
horizon 9

moduli 6
Euclidean 6
Born-Infeld 6
Carroll 4

Gibbons-Manton metric for N monopoles
Attractor-mechanism for BHs (flux compacts)
Today’s talk will be about a famous moduli space: that of Calabi-Yau manifolds and its associated solitons: \textbf{wrapped D-branes}.

\[ \times \mathbb{R}^{1,3} \cong \text{Our world?} \]

\[ \text{Moduli} (\lambda) : \text{massless scalar fields; metric determines eff. action} \]

\[ \text{D-branes: solitons with } \lambda \text{- dependent mass and charge} \]
Recent progress in **susy field theories** has opened a new avenue for the solution of these problems.

Computing $g_{I\bar{J}}(\lambda)$ and $M_{S}(\lambda)$ is time-honored problem in physics and mathematics.

They are related to **2D Weyl anomalies**, which shows in turn that they can be computed by the technique of **localization**.

Pestun 2007

....

Benini, Cremonesi 1206.2356
Doroud, Gomis, Le Floch, Lee 1206.2606
Based on dictionary:

**Target space**
- Calabi-Yau
- moduli $\lambda^I$
- moduli space
- metric
- wrapped brane
- mass

**Worldsheet**
- $N=(2,2)$ SCFT
- marginal deformations
- superconformal manifold
- Zamolodchikov metric
- boundary conditions $\Omega$
- bnry degeneracy $g^\Omega$

Affleck, Ludwig

New way to compute r.h.s.
More precisely:

This CY moduli space is known to **factorize locally**: but see arXiv:1611.03101 Gomis, Komargodski, Ooguri, Seiberg, Wang

\[ \mathcal{M}_c \times \mathcal{M}_{tc} \]

$\mathcal{M}_c$ complex structure \((c,c)\)

$\mathcal{M}_{tc}$ Kähler moduli \((c,a)\)

What is the metric on this moduli space?
Strong constraints from $\mathcal{N} = 2$ supersymmetry of type-II string theory compactified on CY3:

$IIA: \ h^{1,1} \ \text{vector} \quad h^{2,1} + 1 \ \text{hyper}$

$IIB: \ h^{2,1} \ \text{vector} \quad h^{1,1} + 1 \ \text{hyper}$

special Kähler $\uparrow$ quaternionic

The string coupling is a hyper, while the volume is a Kähler modulus

metric on complex-structure m.s. is \textbf{classical}

metric on Kähler m.s. has \textbf{instanton} corrections

Gromov-Witten invariants
It has been shown that the Kähler m.s. metric is computed by the **partition function on the 2-sphere**

\[
Z(S^2) = \left(\frac{r}{r_0}\right)^{c/3} e^{-K(\lambda, \bar{\lambda})}
\]

Conjectured by Jockers, Kumar, Lapan, Morrison, Romo (1208.6244)

Shown with help of $\mathbb{tt}^*$ eqns by Gomis + Lee (1210.6022)

Last year: Gomis, Komargodski, Hsin, Schwimmer, Seiberg, Theisen (1509.08511)
gave an elegant new proof based on a "new kind" of **Weyl anomaly**

Osborn ‘91
This is a powerful circle of ideas with non-trivial corollaries and generalizations to higher D:

- Localization → SUSY → Curved-space partition functions → SUSY → Anomalies → Metric of moduli space
We show in particular that the hemisphere partition function computes the second important piece of geometric data:

The central charge $C^\Omega(\lambda)$, and the mass of CY D-branes.

Honda + Okuda 1308.2217
Hori + Romo 1308.2438
Sugishita + Terashima 1308.1973
Rest of this talk:

2  Superconformal manifolds & Anomalies

3  Corollaries for $\mathcal{Z}(S^2)$ and CY moduli space

4  Extension to Boundaries and D-brane mass/charge

5  Summary
2. "Mixed" super-Weyl anomalies

Anomalies imply non-conservation in correlation functions:

$$\langle \partial_\mu j^\mu \mathcal{O}_1(p_1) \cdots \mathcal{O}_n(p_n) \rangle \neq 0$$

When r.h.s. proportional to momenta: non-conservation only in presence of **spacetime-dependent background fields**

e.g.

$$\stackrel{U(1)_A}{G} \quad \begin{array}{c} \Rightarrow \quad \partial_\mu j^\mu_A = F \wedge F \\
\text{axial charge violated by instantons, cf 't Hooft matching conditions} \end{array}$$
For chiral anomalies: background is gauge or gravitational
For trace (Weyl) anomaly, can be **exactly-marginal couplings**:

In 2D the 2-point function of marginal operators reads:

\[
\langle \mathcal{O}_I(z) \bar{\mathcal{O}}_J(w) \rangle = g_{IJ} \mathcal{R} \frac{1}{|z - w|^4} = g_{IJ} \frac{1}{2} (\partial \bar{\partial})^2 \left[ \log(|z - w|^2 \mu^2) \right]^2
\]

**Zamolodchikov metric**

**anomaly**

**differential regularization of distribution**
Turn on space-dependent couplings $\lambda^I$:

$$\frac{\partial Z}{\partial \log \mu} \sim \int_z \int_w \lambda^I(z) \bar{\lambda}^\bar{J}(w) \frac{\partial}{\partial \log \mu} \langle \mathcal{O}_I(z) \bar{\mathcal{O}}_{\bar{J}}(w) \rangle \sim \int \partial_\mu \lambda^I \partial^\mu \bar{\lambda}^\bar{J} g_{I\bar{J}}$$

Anomaly is **invisible for constant couplings**. But supersymmetry relates it to a term that does not vanish when $\partial_\mu \lambda^I = 0$

Gomis et al (1509.08511)

This term can be removed by non-susy local counterterm; **but SUSY gives it universal meaning**
To \( \mathcal{N} = (2, 2) \) SCFTs have \( U(1)_V \times U(1)_A \) R-symmetry.

In computing the anomaly we choose to preserve the \textbf{vector-like} symmetry, so we must couple it to the \( \mathcal{N} = 2 \) supergravity in which this symmetry is gauged by a field \( V^\mu \)

Closset + Cremonesi (1404.2636)

In superconformal gauge:

\[
g_{\mu\nu} = e^{2\sigma} \eta_{\mu\nu}, \quad V^\mu = \epsilon^{\mu\nu} \partial_\nu a
\]

Classically \( \sigma \) and \( a \) decouple, but in the quantum theory they don't due to the \textbf{Weyl} and \textbf{axial} anomalies.
Supersymmetry places these fields in a twisted-chiral multiplet

\[ \Sigma(y^\mu) = (\sigma + i\alpha) + \theta^+ \bar{\chi}_+ + \bar{\theta}^- \chi_- + \theta^+ \bar{\theta}^- w \]

with components functions of

\[ y^\pm = x^\pm \mp i\theta^\pm \bar{\theta}^\pm \]

The tc field obeys

\[ \bar{D}_+ \Sigma = D_- \Sigma = 0 \]

where

\[ D_\pm = \frac{\partial}{\partial \theta^\pm} - i\bar{\theta}^\pm \partial_\pm, \quad \bar{D}_\pm = -\frac{\partial}{\partial \bar{\theta}^\pm} + i\theta^\pm \partial_\pm . \]

It is useful to also promote the marginal couplings to vevs of tc fields

\[ \Lambda^I = \lambda^I(y^\pm) + \cdots , \quad \bar{\Lambda}^I = \bar{\lambda}^I(\bar{y}^\pm) + \cdots \]

so as to make the susy of the anomaly manifest.
The anomaly \( iA(\delta \Sigma) := \delta \Sigma \log \mathcal{Z}_V(M) \) is the susy invariant

\[
A_{\text{closed}} := A^{(1)} + A^{(2)} = \frac{1}{4\pi} \int_M d^2 x \int d^4 \theta \left[ \frac{c}{6} (\delta \Sigma \tilde{\Sigma} + \delta \tilde{\Sigma} \Sigma) - (\delta \Sigma + \delta \tilde{\Sigma}) K(\Lambda, \bar{\Lambda}) \right]
\]

Gomis et al (1509.08511)

This obeys **Wess-Zumino** consistency \( \delta \Sigma A(\delta \Sigma') - \bar{\delta} \Sigma A(\delta \Sigma) = 0 \)

and can be integrated with the result:

\[
\log \mathcal{Z}_V \supset \frac{i}{4\pi} \int_M d^2 x \int d^4 \theta \left[ \frac{c}{6} \Sigma \tilde{\Sigma} - (\Sigma + \tilde{\Sigma}) K \right].
\]
Expand in components:

\[ A^{(1)} = -\frac{c}{12\pi} \int_M d^2x \left[ \delta\sigma \Box \sigma + \delta a \Box a + \frac{1}{2} (\delta w \, \bar{w} + \delta \bar{w} \, w) + \partial^\mu b_\mu^{(1)} \right] + \text{fermions} , \]

\[ A^{(2)} = -\frac{1}{2\pi} \int_M d^2x \left[ \delta\sigma (\partial_\mu \lambda^I \partial^\mu \bar{\lambda}^J) \partial_I \partial_J K - \frac{1}{2} K \Box \delta\sigma - (\partial^\mu \delta a) K_\mu + \partial^\mu b_\mu^{(2)} \right] \]

where \( K_\mu := \frac{i}{2} (\partial_I K \partial_\mu \lambda^I - \partial_I K \partial_\mu \bar{\lambda}^I) \)

(Cohomologically) non-trivial, real anomalies

Variation of \textbf{local invariant counterterm}

\[ \sim \int \sqrt{g} R^{(2)} K (\lambda, \bar{\lambda}) \]
The first term in $A^{(2)}$ is the **scale anomaly in the 2-point function**

as follows from $\delta \sigma = -\delta \log \mu$, $\partial \bar{\partial} \log |z|^2 = \pi \delta^{(2)}(z)$

and $\partial_I \partial_{\bar{J}} K = g_{I\bar{J}}$

The non-vanishing term for constant couplings is the **red** one

It could be removed by **change of scheme** in bosonic theory, but supersymmetry relates it to the non-trivial **blue** terms!

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Similar remark made previously for 4D Casimir energy by Assel, Cassani, Di Pietro, Komargodski, Lorenzen, Martelli 1503.05537
3. Some corollaries

Integrating the anomaly for constant couplings gives

$$\int_{S^2} K \, \Box \sigma = -4\pi K \implies Z^E_V(S^2) = \left( \frac{r}{r_0} \right)^{c/3} e^{-K(\lambda, \bar{\lambda})}$$

so the 2-sphere free energy computes the Kähler potential on the SCFT2 moduli space (both chiral and twisted chiral)

**A puzzle**

$Z^E_V(S^2)$ not invariant under Kähler-Weyl transformations

$$K'(\lambda, \bar{\lambda}) = K(\lambda, \bar{\lambda}) + H(\lambda) + \bar{H}(\bar{\lambda})$$
Resolution

The difference amounts to change of renormalization scheme:

\[ \Delta_{KW} A^{(2)} = -\frac{1}{4\pi} \int_M d^2x \int d^4\theta (\delta \Sigma + \delta \bar{\Sigma}) H + c.c. \]

\[ = -\frac{1}{4\pi} \int_M d^2x \int d\theta^+ d\bar{\theta}^- (\bar{D}+D- \delta \bar{\Sigma}) H + \int_M d^2x (\partial^\mu Y_\mu) + c.c. \]

\[ R = \bar{D}+D- \Sigma = -\bar{w} + \theta^+ \bar{\theta}^- \partial_+ \partial_- (\sigma - ia) + \cdots \]

curvature superfield

So local, susy and diffeo-invariant counterterm compensates the Kähler–Weyl (gauge) transformation!
An interesting conjecture

Gomis et al. (1509.08511)

If the moduli space had non-vanishing Kähler class one could pick $\lambda^I(x)$ such that $S^2 \to \mathcal{M}$ is non-trivial 2-cycle.

Then there would be no global renormalization scheme!

Way out: Moduli space has Kähler class $= 0$
Consider half space:

\[ x^1 \leq 0 \quad \text{and} \quad x^1 \geq 0 \]

boundary condition \( \Omega \)

One-point functions of marginal operators:

\[
\langle \mathcal{O}_I(x) \rangle_\Omega = d_I^\Omega \mathcal{R} \frac{1}{|x_1|^2} = d_I^\Omega \partial_1^2 [\Theta(-x^1) \log |x^1 \mu|] \Omega
\]
For susy-preserving (B-type) boundary conditions, the 1pt-function coefficients are not independent. They are related to a holomorphic boundary charge $c^\Omega(\lambda)$

$$\frac{c_I^\Omega}{c^\Omega} = \partial_I (K + \log c^\Omega)$$

Ooguri, Oz, Yin '96

**Argument:** vacuum projection of boundary state

$$\Pi_{\text{vac}} |\Omega\rangle := c^\Omega |0\rangle_{\text{RR}} + \sum_I c_I^\Omega |I\rangle_{\text{RR}}$$

is flat section of the improved connection $\nabla - C$ on moduli space

structure constants of chiral ring
Our result: prove these relations from Weyl–Osborn anomaly, and show that hemisphere p.f. computes bnry charge

\[ Z_+(D^2) = \left( \frac{r}{r_0} \right)^{c/6} c^\Omega(\lambda), \quad Z_-(D^2) = \left( \frac{r}{r_0} \right)^{c/6} c^\Omega(\bar{\lambda}). \]

Under Kähler Weyl transformations

\[ c^\Omega \rightarrow c^\Omega e^F \]

The **boundary entropy** is the scheme-independent combination

\[ g^\Omega = \frac{|c^\Omega|}{e^{-K/2}} = \sqrt{\frac{Z_+(D^2)Z_-(D^2)}{Z(S^2)}} \]
In string-theory compactifications, $g^\Omega$ and $c^\Omega$ are the mass and RR charge of the 1/2 BPS D-brane states dyons in field-theory limits.

These follow from worldsheet anomalies!
3 steps in calculation:

Take into account the divergence terms in $A_{\text{closed}}$

\[
b^{(1)}_{\mu} = \frac{1}{4} (\partial_\mu \delta \sigma) \sigma - \frac{3}{4} \delta \sigma \partial_\mu \sigma + \frac{1}{4} (\partial_\mu \delta a) a - \frac{3}{4} \delta a \partial_\mu a
\]

\[
b^{(2)}_{\mu} = \frac{1}{4} (\partial_\mu \delta \sigma) K - \frac{1}{4} \delta \sigma \partial_\mu K.
\]

Add ‘minimal’ boundary term needed for susy

Extra boundary-superinvariant additions using formalism of boundary superspace
Consider the D-term:

\[
\int_{M} d^{2}x \int d^{4}\theta \mathcal{S} = \int_{M} d^{2}x [\mathcal{S}]_{\text{top}}
\]

The type-B susy generator is

\[
\mathcal{D}_{\text{susy}} = \epsilon (Q_{+} + Q_{-}) - \bar{\epsilon} (\bar{Q}_{+} + \bar{Q}_{-})
\]

where 

\[
Q_{\pm} = \frac{\partial}{\partial \theta_{\pm}} + i\bar{\theta}^{\pm} \partial_{\pm}, \quad \bar{Q}_{\pm} = -\frac{\partial}{\partial \theta_{\pm}} - i\theta^{\pm} \partial_{\pm}
\]

The transformation of the D-term is a total derivative

\[
\Delta_{\text{susy}}[\mathcal{S}]_{\text{top}} = \int d^{4}\theta \mathcal{D}_{\text{susy}}\mathcal{S} = i\epsilon \int d^{4}\theta (\bar{\theta}^{+} \partial_{+}\mathcal{S} + \bar{\theta}^{-} \partial_{-}\mathcal{S}) + \text{c.c.}
\]

We want to write as the susy transformation of a boundary term.
Standard manipulations give:

\[ \Delta_{\text{susy}} [S]_{\text{top}} = -\Delta_{\text{susy}} ( \partial_1 [S]_{\text{bnry}} ) + \partial_0 Y \]

with

\[ [S]_{\text{bnry}} = -\frac{i}{2} ( [S]_{\theta^+ \bar{\theta}^-} - [S]_{\theta^- \bar{\theta}^+} ) - \frac{1}{4} \partial_1 [S]_{\theta} \]

so that

\[ I_D(S) := \int d^2 x [S]_{\text{top}} + \int d^0 x [S]_{\text{bnry}} \]

is our susy-invariant standard completion.

For the case of interest, the integrated superfield is \( \delta S \)

with

\[ S = \frac{1}{4\pi} \left[ \frac{c}{6} \Sigma \bar{\Sigma} - (\Sigma + \bar{\Sigma})K \right] \]
Boundary superspace

\[ x^+ = x^- , \quad \theta \equiv e^{-i\beta} \theta^+ = e^{i\beta} \theta^- , \quad \bar{\theta} \equiv e^{i\beta} \bar{\theta}^+ = e^{-i\beta} \bar{\theta}^- \]

Restrictions of bulk superfields, e.g.

\[ \Sigma|_{\partial M} = \sigma + ia + \theta \bar{\chi}_+ + \bar{\theta} \chi_- + \theta \bar{\theta} [w - i \partial_1 (\sigma + ia)] \]

Usual D-term and F-term integrals of bny superfields are invariant

**WZ-consistency, locality and parity covariance** leads to ansatz for boundary-superinvariant contribution to anomaly:

\[
\int dx^0 \left[ \mathcal{B} \right]_{\theta \bar{\theta}} \quad \text{where} \quad \mathcal{B} = \frac{i}{8\pi} \left[ \frac{c}{12} (\Sigma^2 - \bar{\Sigma}^2) + \Sigma \tilde{G}^\Omega (\Lambda, \bar{\Lambda}) - \bar{\Sigma} G^\Omega (\bar{\Lambda}, \Lambda) \right] \bigg|_{\partial M}
\]

and reality condition \( G^\Omega (\bar{\Lambda}, \Lambda) = [G^\Omega (\Lambda, \bar{\Lambda})]^* \)
Collecting everything:

\[
A_{\text{open}} = \int_M d^2 x \, [\delta S]_{\text{top}} + \int_{\partial M} dx^0 \, ([\delta S]_{\text{bnry}} + [\delta B]_{\theta \bar{\theta}})
\]

where

\[
S = \frac{1}{4\pi} \left[ \frac{c}{6} \Sigma \bar{\Sigma} - (\Sigma + \bar{\Sigma}) K \right]
\]

\[
[S]_{\text{bnry}} = -\frac{i}{2} \left( [S]_{\theta + \bar{\theta} -} - [S]_{\theta - \bar{\theta} +} \right) - \frac{1}{4} \partial_1 [S]_0
\]

\[
B = \frac{i}{8\pi} \left[ \# \frac{c}{12} (\Sigma^2 - \bar{\Sigma}^2) + \bar{\Sigma} G^\Omega (\Lambda, \bar{\Lambda}) - \Sigma G^\Omega (\bar{\Lambda}, \Lambda) \right] \bigg|_{\partial M}
\]

central-charge anomaly

Weyl-Osborn anomaly

cf Polchinski; Solodukhin for higher D
Susy Ward identity: \( \langle \int \delta L_{\text{sugra}} \int \delta L_{\text{SCFT}} \rangle = 0 \) \quad \text{if} \quad \delta \tilde{\Sigma} = \tilde{\Lambda}^{I} = 0

\implies \text{no terms proporto} \quad \delta \Sigma \Lambda^{I}

\implies \quad G^{\Omega}(\lambda, \bar{\lambda}) = K(\lambda, \bar{\lambda}) + 2 \log c^{\Omega}(\lambda)

\textbf{Kähler-Weyl} covariance (up to local counterterms) requires

\( c^{\Omega} := e^{h^{\Omega}} \) \quad \text{section of holomorphic line bundle}

\begin{align*}
K &\to K + H + \bar{H} \\
\h^{\Omega} &\to \h^{\Omega} - H
\end{align*}
final ingredient: susy hemisphere

\[ A_{\text{open}} \supset \delta \left\{ -\frac{1}{4\pi} \int d^2 x \left[ \Box (\sigma - ia) h^\Omega + \Box (\sigma + ia) \bar{h}^\Omega \right] + \frac{i}{4\pi} \int dx^0 \left[ \bar{w} h^\Omega - w \bar{h}^\Omega \right] \right\} \]

integrated anomaly subtracted so as to vanish for infinitesimal
disk depends only the holomorphic boundary charge, plus the
auxiliary field of the metric.

Seiberg, Festuccia 1105.0689
Killing-spinor equations imply

\[ w = 2i \frac{\zeta^-}{\zeta^+} \partial_z (\sigma + ia + \log \zeta^-) = 2i \frac{\zeta^+}{\zeta^-} \partial_z (\sigma + ia + \log \zeta^+) , \]

\[ \bar{w} = -2i \frac{\zeta^+}{\zeta^-} \partial_{\bar{z}} (\sigma - ia + \log \zeta^+) = -2i \frac{\bar{\zeta}^-}{\zeta^+} \partial_{\bar{z}} (\sigma - ia + \log \bar{\zeta}^-) . \]

where the unbroken superconformal symmetries are

\[ \epsilon_+ = \epsilon \zeta^- (z) , \quad \epsilon_- = -\epsilon \zeta^+ (\bar{z}) , \quad \bar{\epsilon}_+ = \bar{\epsilon} \bar{\zeta}^- (z) , \quad \bar{\epsilon}_- = -\bar{\epsilon} \bar{\zeta}^+ (z) \]

Two solutions for hemisphere with B-type bnr condition:

\begin{align*}
\text{(+:)} & \quad \zeta^- = 1, \quad \zeta^+ = \bar{z}, \quad \bar{\zeta}^- = z, \quad \bar{\zeta}^+ = 1, \\
\text{(-:)} & \quad \zeta^- = z, \quad \zeta^+ = -1, \quad \bar{\zeta}^- = 1, \quad \bar{\zeta}^+ = -\bar{z}
\end{align*}
Supersymmetric hemispheres with B-type bndy condition:

\[ \sigma = - \log(1 + z\bar{z}) + \text{constant}, \quad a = 0 \]

\[ (+): \quad w = \bar{w} = -\frac{2i}{1 + z\bar{z}}, \quad (-): \quad w = \bar{w} = \frac{2i}{1 + z\bar{z}} \]

which implies

\[ Z_+ (D^2, \Omega) = Z_0 \, c^\Omega (\lambda), \quad Z_- (D^2, \Omega) = Z_0 \, c^\Omega (\bar{\lambda}). \]

\[ \text{qed} \]
Computed the super-Weyl anomaly for $\mathcal{N} = (2, 2)$ models on a surface with boundary generalizing the result of Gomis, Komargodski, Hsin, Schwimmer, Seiberg, Theisen (1509.08511)

Not only the Kähler potential but also the brane charge & mass are given by an (‘Osborn-type’) anomaly. They can be computed by localization.

Extension to higher dimensions and other co-dimension defects
Many thanks to the organizers and congratulations to Gary!