Arbitrary order discontinuous Galerkin methods for hyperbolic balance laws

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Outline

Hyperbolic balance laws

- Hyperbolic systems of conservation laws with source terms (also called balance laws):

  $$U_t + f(U)_x = s(U, x)$$

  Steady state solution, i.e. solution of $f(U)_x = s(U, x)$.

- Standard numerical schemes usually fail to capture the steady state well and introduce spurious oscillations. The grid must be extremely refined to reduce the size of these oscillations.

- Well-balanced methods are developed to reduce the unnecessarily refined mesh. They are specially designed to preserve exactly these steady-state solutions up to machine error with relatively coarse meshes.

A typical example of balance laws

Shallow water equations (SWEs) with a non-flat bottom topography:

$$h_t + (hu)_x = 0$$
$$(hu)_t + \left(hu^2 + \frac{1}{2}gh^2\right)_x = -ghb_x$$

- h : water height; u : velocity;
  b : bottom topography; g : gravitational constant.
- Still water at rest steady state:

  $$u = 0 \qquad \text{and} \qquad h + b = const.$$

  Moving water steady state:

  $$hu = const \qquad \text{and} \qquad u^2/2 + g(h+b) = const.$$
- Extensive well-balanced methods have been developed in the past two decades.

Euler equations with a gravitational potential

Euler equations with a source term due to the static gravitational field:

$$\rho_t + \nabla \cdot (\rho \mathbf{u}) = 0,$$
$$(\rho \mathbf{u})_t + \nabla \cdot (\rho \mathbf{u} \otimes \mathbf{u} + p \mathbf{I}_d) = -\rho \nabla \phi,$$
$$E_t + \nabla \cdot ((E + p)\mathbf{u}) = -\rho \mathbf{u} \cdot \nabla \phi,$$

- ρ : fluid density; $\mathbf{u}$: velocity; p : pressure;
  $E = \frac{1}{2}\rho u^2 + p/(\gamma - 1)$: non-gravitational energy.
- $\phi = \phi(\mathbf{x})$: time independent gravitational potential.
  A simple example is: $\phi_z = g$.
- The hydrostatic balance with a zero velocity:

$$\rho = \rho(x), \qquad u = 0, \qquad \nabla p = -\rho \nabla \phi,$$

where the flux produced by the pressure balances the gravitational source.

Steady state solutions

- Two important special steady state are the constant entropy (isentropic/polytropic) and constant temperature (isothermal) hydrostatic equilibrium states
- Isothermal equilibrium: For an ideal gas, we have $p = \rho RT$. The equilibrium (with constant temperature T_0) becomes

$$\rho = \rho_0 \exp\left(-\frac{\phi}{RT_0}\right), \qquad u = 0, \qquad p = RT_0\rho = p_0 \exp\left(-\frac{\phi}{RT_0}\right).$$

- Polytropic equilibrium:

$$p = K\rho^\gamma,$$

which will lead to the form of

$$\rho = \left(\frac{\gamma-1}{K\gamma}(C-\phi)\right)^{\frac{1}{\gamma-1}}, \qquad \mathbf{u} = 0, \qquad p = \frac{1}{K^{\frac{1}{\gamma-1}}}\left(\frac{\gamma-1}{\gamma}(C-\phi)\right)^{\frac{\gamma}{\gamma-1}},$$

or equivalently,

$$h + \phi = const,$$

where $h = e + p/\rho$ is the specific enthalpy and e is the specific internal energy.

Discontinuous Galerkin methods

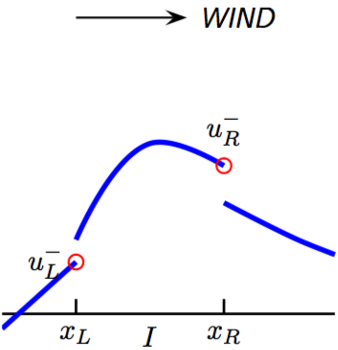
Discontinuous Galerkin (DG) methods

To solve

$$u_t + u_x = 0$$
$$\Downarrow$$
$$u_t v + u_x v = 0$$
$$\Downarrow$$
$$\int_I u_t v dx + \int_I u_x v dx = 0, \qquad \text{on } I = (x_L, x_R)$$
$$\Downarrow$$
$$\int_I u_t v dx - \int_I u v_x dx + uv(x_R) - uv(x_L) = 0, \qquad \text{on } I = (x_L, x_R)$$

Weak formulation!

DG methods



Find $u \in V_h$= piecewise polynomial space, s.t. for any $v \in V_h$,

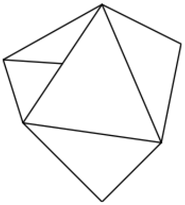
$$\int_I u_t v \, dx - \int_I u v_x \, dx + \hat{u} v^-(x_R) - \hat{u} v^+(x_L) = 0.$$

Numerical Fluxes

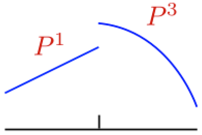
$$\hat{u} = u^-.$$

Advantage of DG methods

- Use of Finite Volume methods framework, convection-dominated problems.
- Flexibility with geometry and mesh (hanging nodes)



- h-p refinement, polynomials of different degrees in different elements, even non-polynomial basis.



- Highly parallelizable, good for high performance computers (high computation intensity and less communications).

Literature of DG methods

- First introduced by Reed and Hill (1973) for neutron transport equation, Lesaint and Raviart (1974).
- For elliptic problems: Babuska et al. (1973), Baker (1977), Wheeler (1978), Arnold (1979), Arnold et al. (2002), Cockburn et al. (2008) $\cdots$
- For parabolic problems: Bassi and Rebay (1997), Cockburn and Shu (1998) $\cdots$
- Runge Kutta DG (RKDG) for hyperbolic conservation problems: Cockburn and Shu (1989-1998) $\cdots$

The main objective

Develop high order accurate well-balanced discontinuous Galerkin schemes for the shallow water equations and Euler equations with source terms, which have the key advantage

- High order accuracy;
- Well-balanced for the steady state solutions;
- Positivity-preserving for the water height of SWEs;
- Good resolution for smooth and discontinuous solutions.

We expect well-balanced methods to be efficient for time dependent problems, which are small perturbation of the steady state solutions.

Euler equations with a gravitational potential

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$$\begin{aligned}
&\rho_t + \nabla \cdot (\rho \mathbf{u}) = 0, \\
&(\rho \mathbf{u})_t + \nabla \cdot (\rho \mathbf{u} \otimes \mathbf{u} + p \mathbf{I}_d) = -\rho \nabla \phi, \\
&E_t + \nabla \cdot ((E + p)\mathbf{u}) = -\rho \mathbf{u} \cdot \nabla \phi,
\end{aligned}$$

- ρ : fluid density; $\mathbf{u}$: velocity; p : pressure;
  $E = \frac{1}{2}\rho u^2 + p/(\gamma - 1)$: non-gravitational energy.
- $\phi = \phi(\mathbf{x})$: time independent gravitational potential.
  A simple example is: $\phi_z = g$.
- The hydrostatic balance with a zero velocity:

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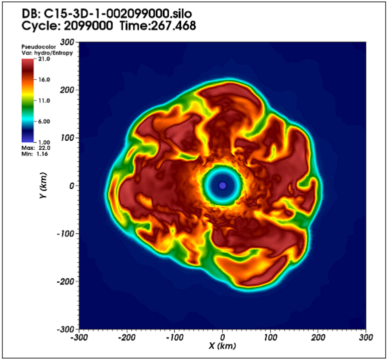
where the flux produced by the pressure balances the gravitational source.

Motivation and existing approaches

- Motivation: core-collapse supernova simulation



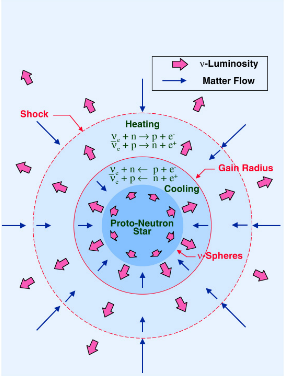


Figure: Image credit: Left: Endeve, Mezzacappa et al. (ORNL); Right: TeraScale Supernova Initiative.

- Many astrophysical problems involve the hydrodynamical evolution in a gravitational field. It is essential to correctly capture the effect of gravitational force in the simulations.

Motivation and existing approaches

- Other applications:
  1. Stellar "evolution": Stars evolve mostly quietly, very close to a hydrostatic state.
  2. Waves in stellar atmospheres: The wave amplitude may be much smaller when compared to the stratification from gravity...
  3. Atmospheric flows: Atmospheric motions happen on a hydrostatic background.

Motivation and existing approaches

- Other applications:
  1. Stellar "evolution": Stars evolve mostly quietly, very close to a hydrostatic state.
  2. Waves in stellar atmospheres: The wave amplitude may be much smaller when compared to the stratification from gravity...
  3. Atmospheric flows: Atmospheric motions happen on a hydrostatic background.
- Improper treatment of the gravitational force can introduce large spurious oscillations, unless the grid is extremely refined.
- Some attempts in designing well-balanced methods for the Euler equations.

  LeVeque and Bale 1998
  Botta, Klein, Langenberg and Lützenkirchen 2004
  Xu and his collaborators 2007, 2010, 2011
  Käppeli and Mishra 2014, 2016
  Chandrashekar, Klingenberg, Puppo et. al. 2015, 2018
  Chertock, Cui, Kurganovz, Özcan and Tadmor 2015
  Xing et al. 2013, 2015, 2016
  ...

Steady state solutions

- The hydrostatic balance with a zero velocity:

$$\rho = \rho(x), \qquad u = 0, \qquad p_x = -\rho\phi_x,$$

where the flux produced by the pressure balances the gravitational source. Two important special steady state are the constant entropy (isentropic/polytropic) and constant temperature (isothermal) hydrostatic equilibrium states

- Isothermal equilibrium: For an ideal gas, we have $p = \rho RT$. The equilibrium (with constant temperature T_0) becomes

$$\rho = \rho_0 \exp\left(-\frac{\phi}{RT_0}\right), \quad u = 0, \quad p = RT_0\rho = p_0 \exp\left(-\frac{\phi}{RT_0}\right).$$

A special case with a linear gravitational potential field: $\phi_x = g$ is:

$$\rho = \rho_0 \exp(-g\rho_0 x/p_0), \qquad \mathbf{u} = 0, \qquad p = p_0 \exp(-g\rho_0 x/p_0).$$

Steady state solutions

- Polytropic equilibrium:

$$p = K\rho^\gamma,$$

which will lead to the form of

$$\rho = \left(\frac{\gamma-1}{K\gamma}(C-\phi)\right)^{\frac{1}{\gamma-1}}, \qquad \mathbf{u}=0, \qquad p = \frac{1}{K^{\frac{1}{\gamma-1}}}\left(\frac{\gamma-1}{\gamma}(C-\phi)\right)^{\frac{\gamma}{\gamma-1}},$$

or equivalently,

$$h + \phi = const,$$

where $h = e + p/\rho$ is the specific enthalpy and e is the specific internal energy.

A special case with a linear gravitational potential field: $\phi_x = g$ is:

$$p = p_0^{\frac{1}{\gamma-1}}\left(p_0 - \frac{\gamma-1}{\gamma}g\rho_0 x\right)^{\frac{\gamma}{\gamma-1}}, \qquad u=0, \qquad \rho = \rho_0\left(\frac{p}{p_0}\right)^{\frac{1}{\gamma}}.$$

First and second order finite volume well-balanced methods for polytropic equilibrium are designed by Käppeli and Mishra (2014).

Key idea

- Discretize the source terms using an approximation consistent with that of approximating the flux derivative terms.
- This idea has been used to design well-balanced methods for the shallow-water equations by us.
- The simple steady state

$$\rho = c\exp(-gx), \qquad u = 0, \qquad p = c\exp(-gx),$$

with $\phi_x = g$ will be used as an example to illustrate the idea.

Equivalent form

- Rewrite the equations as

$$\rho_t + (\rho u)_x = 0$$
$$(\rho u)_t + (\rho u^2 + p)_x = \frac{\rho}{\exp(-gx)}(\exp(-gx))_x$$
$$E_t + ((E+p)u)_x = -\rho u g,$$

  Purpose: introduce the derivative term in the source term, which can be treated in the similar way as the flux term.

- Denote them by

$$U_t + F(U)_x = S(U, \phi).$$

Semi-discrete DG scheme

$$U_t + F(U)_x = S(U, \phi).$$

- The semi-discrete DG scheme

$$\int_{I_j} (U_h)_t v \mathrm{d}x - \int_{I_j} F(U_h) v_x \mathrm{d}x + \hat{F}_{j+\frac{1}{2}} v(x^-_{j+\frac{1}{2}}) - \hat{F}_{j-\frac{1}{2}} v(x^+_{j-\frac{1}{2}}) = \int_{I_j} S v \mathrm{d}x,$$

where

$$\widehat{F}_{j+\frac{1}{2}} = f(U_h(x^-_{j+\frac{1}{2}}, t), U_h(x^+_{j+\frac{1}{2}}, t)),$$

and $f(a_1, a_2)$ is a numerical flux.

- Lax-Friedrichs flux:

$$f(a_1, a_2) = \frac{1}{2}(F(a_1) + F(a_2) - \alpha(a_2 - a_1)).$$

Well-balanced source term approximation

- We first decompose the integral of the source term in the second equation as

$$\int_{I_j} S_2 v \mathrm{d}x = \int_{I_j} \rho \exp(gx) \left(\exp(-gx)\right)_x v \mathrm{d}x = \int_{I_j} \frac{\rho}{b} b_x v \mathrm{d}x$$

$$= \frac{\rho(x_j)}{b(x_j)} \left(b(x^-_{j+\frac{1}{2}}) v(x^-_{j+\frac{1}{2}}) - b(x^+_{j-\frac{1}{2}}) v(x^+_{j-\frac{1}{2}}) - \int_{I_j} b v_x \mathrm{d}x \right) + \int_{I_j} \left(\frac{\rho}{b} - \frac{\rho(x_j)}{b(x_j)} \right) b_x v \mathrm{d}x,$$

where $b(x) = \exp(-gx)$.

- Let $b_h(x)$ be the projection of $b(x)$ into V_h^k, approximate the integral by:

$$\int_{I_j} S_2 v \mathrm{d}x \approx \int_{I_j} \left(\frac{\rho_h}{b_h} - \frac{\rho_h(x_j)}{b_h(x_j)} \right) (b_h)_x v \mathrm{d}x$$

$$+ \frac{\rho_h(x_j)}{b_h(x_j)} \left(\{b_h\}(x_{j+\frac{1}{2}}) v(x^-_{j+\frac{1}{2}}) - \{b_h\}(x_{j-\frac{1}{2}}) v(x^+_{j-\frac{1}{2}}) - \int_{I_j} b_h v_x \mathrm{d}x \right).$$

- Use quadrature rule to approximate the source term in the third equation

$$\int_{I_j} S_3 v \mathrm{d}x \approx \int_{I_j} -(\rho u)_h g v \mathrm{d}x.$$

Well-balanced fix to the numerical flux

- Lax-Friedrichs flux:

$$f(a_1, a_2) = \frac{1}{2}(F(a_1) + F(a_2) - \alpha(a_2 - a_1)).$$

  $\alpha(a_2 - a_1)$ contributes to the numerical viscosity term, but may destroy the well-balanced property at the steady state.

- Well-balanced modification:

$$\hat{F}_{j+1/2} = \frac{1}{2}\left[F\left(U_h(x^-_{j+1/2})\right) + F\left(U_h(x^+_{j+1/2})\right) - \alpha'\left(\frac{U_h(x^+_{j+1/2})}{b_h(x^+_{j+1/2})} - \frac{U_h(x^-_{j+1/2})}{b_h(x^-_{j+1/2})}\right)\right].$$

  To maintain enough artificial numerical viscosity:

$$\alpha' = \alpha \max_x b_h(x),$$

  At the steady state, the numerical flux reduces to

$$\hat{f}_{j+\frac{1}{2}} = \frac{1}{2}\left[f\left(U(x^-_{j+1/2})\right) + f\left(U(x^+_{j+1/2})\right)\right].$$

Main result

Proposition: For the Euler equations with the linear gravitational potential field, the semi-discrete DG methods mentioned above can maintain the original high order accuracy and are well-balanced for the steady state solution.

Proof: At the steady state, we have

$$\rho_h = cb_h, \qquad u = 0, \qquad p_h = cb_h.$$

For the momentum equation, the source term approximation becomes

$$\int_{I_j} S_2 v \mathrm{d}x \approx c\left(\{b_h\}(x_{j+\frac{1}{2}})v(x^-_{j+\frac{1}{2}}) - \{b_h\}(x_{j-\frac{1}{2}})v(x^+_{j-\frac{1}{2}}) - \int_{I_j} b_h v_x \mathrm{d}x\right).$$

Since $u = 0$, the flux term $F_2 = \rho u^2 + p$ reduces to $p = cb$. Its numerical approximation takes the form of

$$\begin{aligned}
&\hat{F}_2(x_{j+\frac{1}{2}})v(x^-_{j+\frac{1}{2}}) - \hat{F}_2(x_{j-\frac{1}{2}})v(x^+_{j-\frac{1}{2}}) - \int_{I_j} F_2 v_x \mathrm{d}x \\
=& c\{b_h\}(x_{j+\frac{1}{2}})v(x^-_{j+\frac{1}{2}}) - c\{b_h\}(x_{j-\frac{1}{2}})v(x^+_{j-\frac{1}{2}}) - \int_{I_j} cb_h v_x \mathrm{d}x.
\end{aligned}$$

General steady state

$$\rho = \rho_0 \exp\left(-\frac{\phi}{RT}\right), \qquad u = 0, \qquad p = RT\rho = RT\rho_0 \exp(-\frac{\phi}{RT}).$$

- We first rewrite the equations:

$$\rho_t + (\rho u)_x = 0,$$
$$(\rho u)_t + \left(\rho u^2 + p\right)_x = RT\rho \exp\left(\frac{\phi}{RT}\right)\left(\exp\left(-\frac{\phi}{RT}\right)\right)_x,$$
$$E_t + ((E+p)u)_x = -\rho u g,$$

- Well-balanced source term approximation:

$$\int_{I_j} S_2 v \mathrm{d}x \approx \int_{I_j} RT\left(\frac{\rho_h}{d_h} - \frac{\rho_h(x_j)}{d_h(x_j)}\right)(d_h)_x v \mathrm{d}x$$
$$+ RT\frac{\rho_h(x_j)}{d_h(x_j)}\left(\{d_h\}(x_{j+\frac{1}{2}})v(x^-_{j+\frac{1}{2}}) - \{d_h\}(x_{j-\frac{1}{2}})v(x^+_{j-\frac{1}{2}}) - \int_{I_j} d_h v_x \mathrm{d}x\right),$$

where $d(x) = \exp\left(-\frac{\phi}{RT}\right)$.

- Well-balanced fix to the numerical flux.

Multi-dimensional Euler equations

- The Euler equations with a static gravitational field are

$$\rho_t + \nabla \cdot (\rho \mathbf{u}) = 0,$$
$$(\rho \mathbf{u})_t + \nabla \cdot (\rho \mathbf{u} \otimes \mathbf{u} + p\mathbf{I}_d) = -\rho \nabla \phi,$$
$$E_t + \nabla \cdot ((E + p)\mathbf{u}) = -\rho \mathbf{u} \cdot \nabla \phi,$$

- Hydrostatic balance:

$$\rho = \rho_0 \exp\left(-\frac{\phi}{RT}\right), \quad \mathbf{u} = 0, \quad p = RT\rho = RT\rho_0 \exp\left(-\frac{\phi}{RT}\right),$$

  with constant temperature T.

- A special case is:

$$\rho = \rho_0 \exp\left(-\rho_0(\mathbf{g} \cdot \mathbf{x})/p_0\right), \quad u = v = 0, \quad p = p_0 \exp\left(-\rho_0(\mathbf{g} \cdot \mathbf{x})/p_0\right),$$

  with a linear gravitational potential field: $\phi(\mathbf{x}) = \mathbf{g} \cdot \mathbf{x}$.

Well-balanced methods

$$\rho = \rho_0 \exp\left(-\frac{\phi}{RT}\right), \qquad \mathbf{u} = 0, \qquad p = RT\rho = RT\rho_0 \exp(-\frac{\phi}{RT}).$$

- We first rewrite the source term:

$$-\rho\nabla\phi = RT\rho \exp\left(\frac{\phi}{RT}\right)\nabla\left(\exp\left(-\frac{\phi}{RT}\right)\right) = \frac{RT\rho}{d}\nabla d.$$

- Well-balanced source term approximation:

$$\int_K S_2 w \,\mathrm{d}x \approx \int_K RT\left(\frac{\rho_h}{d_h} - \frac{\rho_h(\mathbf{x}_K^0)}{d_h(\mathbf{x}_K^0)}\right)\nabla d_h \, w \,\mathrm{d}\mathbf{x}$$
$$+ RT\frac{\rho_h(\mathbf{x}_K^0)}{d_h(\mathbf{x}_K^0)}\left(\sum_{i=1}^{m}\int_{e_K^i}\{d_h(\mathbf{x})\}\nu_K^i w \,\mathrm{d}s - \int_K d_h \nabla w \,\mathrm{d}\mathbf{x}\right),$$

  where $d(x) = \exp\left(-\frac{\phi}{RT}\right)$.

- Well-balanced fix to the numerical flux.

The same technique can be extended to the polytropic equilibrium state.

Different well-balanced approach via hydrostatic reconstructi

Well-balanced methods for the polytropic balance

Key idea:

Decompose the solution into equilibrium and non-equilibrium parts, and treat them differently.

Well-balanced methods for the polytropic balance

Key idea:

Decompose the solution into equilibrium and non-equilibrium parts, and treat them differently.

Components:

- Recovery of well-balanced states;
- Decomposition of the solutions into equilibrium and non-equilibrium parts;
- Numerical fluxes via hydrostatic reconstruction;
- Novel source term approximation.

Use 1D Euler equation as example to demonstrate the algorithm.

Recovery of well-balanced states

Suppose $U(x, t = 0) = U^0(x)$ are in perfect equilibrium, i.e.,

$$u^0(x) = 0, \qquad h(x) + \phi(x) = const\, C.$$

- Initial condition for DG methods is the projection of these to $V_{\Delta x}$.
- Usually L^2 projection is used. But it is difficult to retrieve the constant C from the projected initial condition.
- In our previous FV work for the shallow water equations, we define it as a solution of a nonlinear equation and solve it using Newton's iteration.

Recovery of well-balanced states

Suppose $U(x,t=0)=U^0(x)$ are in perfect equilibrium, i.e.,

$$u^0(x)=0, \qquad h(x)+\phi(x)=const\, C.$$

- DG methods are more flexible.
  We define Projection $P_h^+\omega$ as a projection of $\omega(x)$ into $V_{\Delta x}$:

$$\int_{I_j} P_h^+ \omega v dx = \int_{I_j} \omega v dx,$$

  for any $v \in P^{k-1}$ on I_j, and

$$P_h^+\omega(x^+_{j-\frac{1}{2}}) = \omega(x_{j-\frac{1}{2}}) \quad \text{at the left boundary } x_{j-\frac{1}{2}}.$$

- We can verify this projection is optimal, i.e., $\|P_h^+U(x)-U(x)\| = O(h^{k+1})$, plus we have

$$h(p_h(x_j),\rho_h(x_j)) + \phi_h(x_j) = const\, C,$$

  where $U_h(x)=P_h^+U(x)$, $\rho_h(x)=P_h^+\rho(x)$ etc.

Decomposition into equilibrium and non-equilibrium parts

- Key idea: decompose U_h into the sum of a reference equilibrium state U_h^e and the remaining part U_h^r.
- $U_h^e(x)$?

  Let $h^e(x) = h(p_h(x_j), \rho_h(x_j)) + \phi_h(x_j) - \phi(x)$. Recover $\rho^e(x)$ and $p^e(x)$ from $h^e(x)$. For the ideal gas law, we have the polytropic form

$$\rho^e(x) = \left(\frac{1}{K}\frac{\gamma-1}{\gamma}h^e(x)\right)^{\frac{1}{\gamma-1}},\ u^e(x) = 0,\ p^e(x) = \left(\frac{1}{K}\right)^{\frac{1}{\gamma-1}}\left(\frac{\gamma-1}{\gamma}h^e(x)\right)^{\frac{\gamma}{\gamma-1}}.$$

  We can compute $U^e(x)$, and define $U_h^e(x) = P_h^+ U^e(x)$.
- $U_h^r = U_h(x) - U_h^e(x)$.

  Note that both U_h^e and U_h^r are piecewise polynomials.
- At the polytropic steady state, $U_h^r(x) = 0$.

Well-balanced fluxes (Hydrostatic reconstruction)

- The hydrostatic reconstructed cell boundary values are defined by:

$$U^{*,\pm}_{j+\frac{1}{2}} = U^e\left(h(p_h(x_j), \rho_h(x_j)) + \phi_h(x_j) - \max(\phi^{\pm}_{h,j+\frac{1}{2}})\right) + (U^r_h)^{\pm}_{j+\frac{1}{2}},$$

  In the case of polytropic equilibrium, $U^{*,+}_{j+\frac{1}{2}} = U^{*,-}_{j+\frac{1}{2}}$.

- The left and right fluxes $\widehat{f}^l_{j+\frac{1}{2}}$ and $\widehat{f}^r_{j-\frac{1}{2}}$ are given by:

$$\begin{aligned}
\widehat{f}^l_{j+\frac{1}{2}} &= F(U^{*,-}_{j+\frac{1}{2}}, U^{*,+}_{j+\frac{1}{2}}) + f(U^-_{j+\frac{1}{2}}) - f(U^{*,-}_{j+\frac{1}{2}}), \\
\widehat{f}^r_{j-\frac{1}{2}} &= F(U^{*,-}_{j-\frac{1}{2}}, U^{*,+}_{j-\frac{1}{2}}) + f(U^+_{j-\frac{1}{2}}) - f(U^{*,+}_{j-\frac{1}{2}}).
\end{aligned}$$

Source term approximation

- For $-\int(\rho u)_h(\phi_h)_x v dx$, we apply the Gaussian quadrature rule directly.
- $-\rho_h(\phi_h)_x$ is linear with respect to ρ_h, we have

$$-\int \rho_h \phi_{h,x} v dx = -\int \rho_h^e (\phi_h)_x v dx - \int \rho_h^r (\phi_h)_x v dx,$$

which can be approximated by:

$$-\int \rho_h (\phi_h)_x v dx \approx p^{e,-}_{h,j+\frac{1}{2}} v(x^-_{j+\frac{1}{2}}) - p^{e,+}_{h,j-\frac{1}{2}} v(x^+_{j-\frac{1}{2}}) - \int_{I_j} p_h^e v_x \, \mathrm{d}x - \int_{I_j} \rho_h^r (\phi_h)_x v \, \mathrm{d}x,$$

using the fact that U_h^e is the equilibrium state.

Well-balanced methods for polytropic balance

$$\int_{I_j} \partial_t U^n v dx - \int_{I_j} f(U^n) \partial_x v dx + \widehat{f}^l_{j+\frac{1}{2}} v(x^-_{j+\frac{1}{2}}) - \widehat{f}^r_{j-\frac{1}{2}} v(x^+_{j-\frac{1}{2}}) = \int_{I_j} s(h^n, b) v dx,$$

Propoposition: The DG schemes described above maintain polytropic equilibrium exactly.

Well-balanced methods for polytropic balance

$$\int_{I_j} \partial_t U^n v dx - \int_{I_j} f(U^n)\partial_x v dx + \widehat{f}^l_{j+\frac{1}{2}} v(x^-_{j+\frac{1}{2}}) - \widehat{f}^r_{j-\frac{1}{2}} v(x^+_{j-\frac{1}{2}}) = \int_{I_j} s(h^n, b) v dx,$$

Propoposition: The DG schemes described above maintain polytropic equilibrium exactly.

Remarks

- If there is no gravitation field, i.e., $\phi_x = 0$, our well-balanced DG methods become the traditional DG methods.
- The first order version of our well-balanced methods reduces to the method in (Käppeli and Mishra, JCP 2014).

The same technique can be extended to the isothermal equilibrium state (Li-X, JCP 2018).

Multi-dimensional Euler equations on unstructured meshes

The same four components: (details skipped)

- Recovery of well-balanced states;
- Decomposition of the solutions into equilibrium and non-equilibrium parts;
- Numerical fluxes via hydrostatic reconstruction;
- Novel source term approximation;

Numerical results

- The third order finite element DG schemes are implemented, for the flux and the source terms.
- Time discretization is by the third order TVD Runge-Kutta method:

$$\begin{aligned} U^{(1)} &= U^n + \Delta t \mathcal{F}(U^n) \\ U^{(2)} &= \frac{3}{4}U^n + \frac{1}{4}\left(U^{(1)} + \Delta t \mathcal{F}(U^{(1)})\right) \\ U^{n+1} &= \frac{1}{3}U^n + \frac{2}{3}\left(U^{(2)} + \Delta t \mathcal{F}(U^{(2)})\right), \end{aligned}$$

  where $\mathcal{F}(U)$ is the spatial operator.
- The CFL number is taken as 0.18.

One dimensional polytropic equilibrium solution

- The gravitational force, with $g = \phi_x = 1$, acts in the negative x direction.
- Consider a polytropic equilibrium solution

$$\rho(x) = \left(\rho_0^{\gamma-1} - \frac{1}{K_0}\frac{\gamma-1}{\gamma}gx\right)^{\frac{1}{\gamma-1}}, \quad u(x) = 0, \quad p(x) = K_0\rho(x)^\gamma,$$

in the domain $[0, 2]$, with $\gamma = 5/3,\ \rho_0 = 1,\ p_0 = 1$ and $K_0 = p_0/\rho_0^\gamma$.

Table: L^1 errors for different precisions.

N	Precision	ρ	ρu	E
100	Single	1.01E-6	1.48E-7	8.27E-7
	Double	1.33E-15	1.55E-16	8.75E-16
200	Single	4.53E-6	5.24E-7	2.83E-7
	Double	3.34E-15	5.10E-15	2.67E-16

Perturbation of the equilibrium solution

Impose a small perturbation to the velocity state at the bottom

$$u(0,t) = 10^{-6}\sin(4\pi t)$$

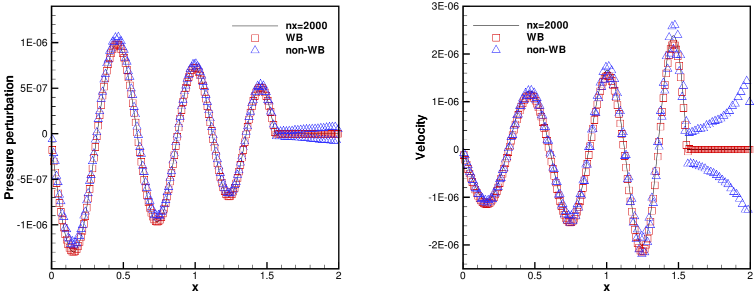


Figure: The pressure perturbations (left) and velocity (right) of a hydrostatic solution with small perturbation. The results of the well-balanced method vs. non-well-balanced method.

Perturbation of the equilibrium solution

Impose a large perturbation to the velocity state at the bottom

$$u(0,t) = 10^{-1}\sin(4\pi t)$$

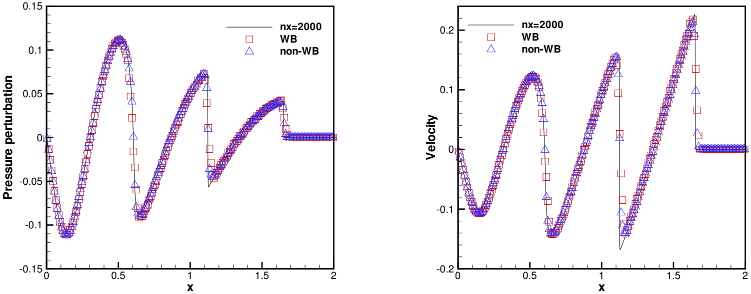


Figure: The pressure perturbations (left) and velocity (right) of a hydrostatic solution with large perturbation. The results of the well-balanced method vs. non-well-balanced method.

One dimensional isothermal equilibrium solution

- The gravitational force, with $g = \phi_x = 1$, acts in the negative x direction.
- Consider an isothermal equilibrium solution

$$\rho_0(x) = p_0(x) = \exp(-x), \qquad \text{and} \qquad u_0(x) = 0.$$

in the domain $[0, 1]$.

Table: L^1 errors for different precisions.

N	Precision	ρ	ρu	E
100	Single	2.38E-7	2.23E-7	4.55E-7
	Double	1.76E-15	1.77E-15	1.24E-15
200	Single	3.13E-7	2.34E-7	4.31E-7
	Double	2.99E-15	1.61E-15	1.84E-15

Perturbation of the equilibrium solution

Impose a small perturbation to the initial pressure state

$$p(x, t = 0) = p_0(x) + \eta \exp(-100(x - 0.5)^2),$$

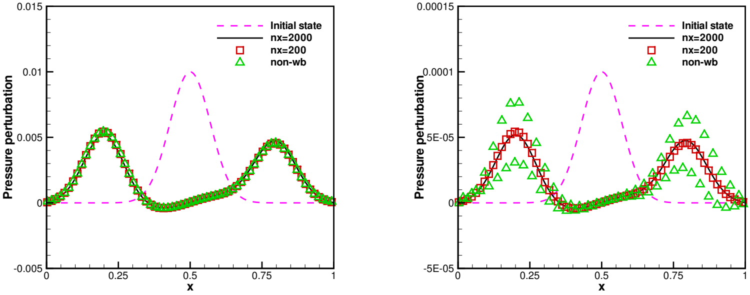


Figure: The pressure perturbation of a hydrostatic solution. The results of the well-balanced method vs. non-well-balanced method. Left: $\eta = 0.01$; Right: $\eta = 0.0001$.

One dimensional gas falling into a fixed external potential

- The gravitational potential has the form of a sine wave,

$$\phi(x) = -\phi_0 \frac{L}{2\pi} \sin \frac{2\pi x}{L},$$

where L is the computational domain length and ϕ_0 is the amplitude.

- Consider an isothermal equilibrium solution

$$\rho = \rho_0 \exp\left(-\frac{\phi}{RT}\right), \qquad u = 0, \qquad p = RT\rho_0 \exp\left(-\frac{\phi}{RT}\right),$$

with a constant temperature T.

- Add a small perturbation to the steady state:

$$\rho = \rho_0 \exp\left(-\frac{\phi}{RT}\right), \quad u = 0,$$
$$p = RT\rho_0 \exp\left(-\frac{\phi}{RT}\right) + 0.001 \exp\left(-10(x-32)^2\right).$$

- We run the simulation with 64 uniform cells for $1,000,000$ time steps.

One dimensional gas falling into a fixed external potential

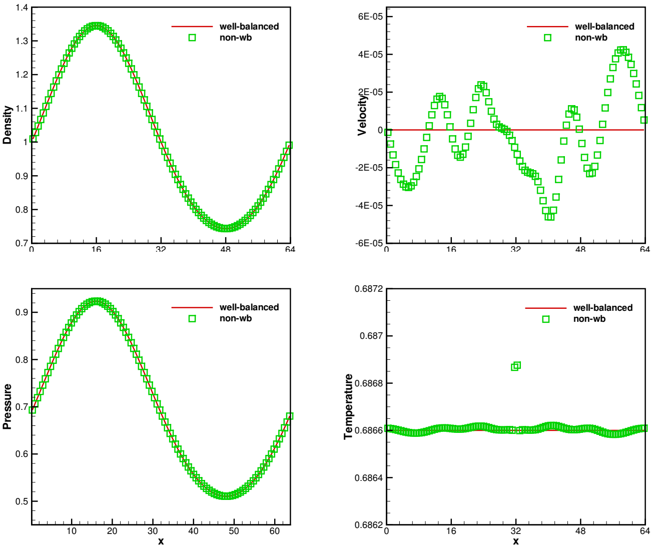


Figure: The numerical solutions of well-balanced method (solid line) and non-well-balanced method (square box, denoted by non-wb)

Two dimensional accuracy test

- Consider a linear gravitational field $\phi_x = \phi_y = 1$, in a computational domain $[0,\, 2] \times [0,\, 2]$.
- A time dependent exact solution

$$
\begin{aligned}
&\rho(x,y,t) = 1 + 0.2\sin(\pi(x+y-t(u_0+v_0))),\\
&u(x,y,t) = u_0, \qquad\qquad v(x,y,t) = v_0,\\
&p(x,y,t) = t(u_0+v_0) - x - y + 0.2\pi\cos(\pi(x+y-t(u_0+v_0))).
\end{aligned}
$$

- The exact solutions are used as the boundary condition. We compute up to $t = 0.1$.

Two dimensional accuracy test

Table: L^1 errors and numerical orders of accuracy for the example (the error of ρv is similar to ρu and is not listed here).

Cells	ρ		ρu		E	
	L^1 error	Order	L^1 error	Order	L^1 error	Order
8×8	1.20E-04		1.09E-04		2.84E-04	
16×16	1.19E-05	3.33	1.13E-05	3.27	3.18E-05	3.16
32×32	1.14E-06	3.39	1.17E-06	3.27	3.61E-06	3.14
64×64	1.35E-07	3.07	1.58E-07	2.89	4.10E-07	3.14
128×128	1.80E-08	2.91	2.15E-08	2.88	4.78E-08	3.10
256×256	2.41E-09	2.90	2.94E-09	2.87	5.93E-09	3.01

Two dimensional polytrope

- An adiabatic gaseous sphere held together by self-gravitation, modeled by the hydrostatic equilibrium

$$\frac{\mathrm{d}p}{\mathrm{d}r} = -\rho \frac{\mathrm{d}\phi}{\mathrm{d}r},$$

and Poisson's equation with $r = \sqrt{x^2 + y^2}$

$$\frac{1}{r^2} \frac{\mathrm{d}}{\mathrm{d}r} \left(r^2 \frac{\mathrm{d}\phi}{\mathrm{d}r} \right) = 4\pi g \rho.$$

- Use the polytropic relation $p = K\rho^\gamma$, assume $\gamma = 2$ to obtain solutions:

$$\rho(r) = \rho_c \frac{\sin(\alpha r)}{\alpha r}, \;\; p(r) = K\rho(r)^2, \tag{1}$$

with $\alpha = \sqrt{\frac{4\pi g}{2K}}$, and the gravitational potential

$$\phi(r) = -2K\rho_c \frac{\sin(\alpha r)}{\alpha r}. \tag{2}$$

The parameters $K = g = \rho_c = 1$ are used.

Small perturbation of the 2D polytrope

Consider a small Gaussian hump perturbations to the initial pressure state

$$p(r) = K\rho(r)^2 + A\exp(-100r^2),$$

where A is taken as 10^{-3}.

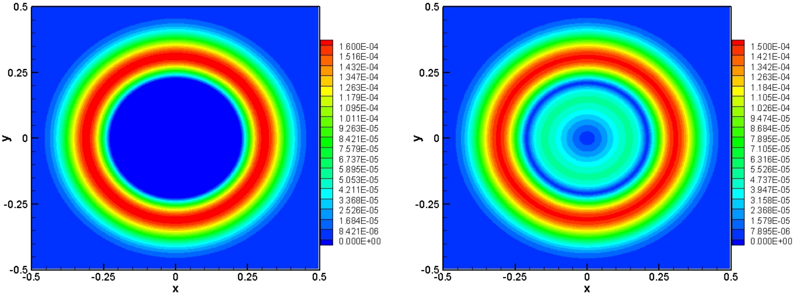


Figure: Well-balanced methods: The contours of the pressure and velocity perturbation of a two dimensional hydrostatic solution with 100×100 cells at $t = 0.2$. Left: pressure p. Right: velocity $\sqrt{u^2 + v^2}$.

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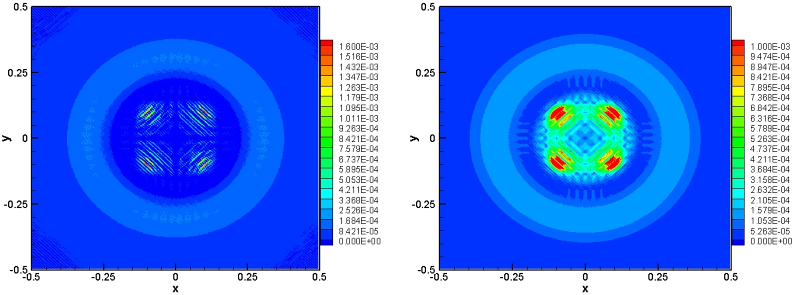


Figure: Non-well-balanced methods: The contours of the pressure and velocity perturbation of a two dimensional hydrostatic solution with 100×100 cells at $t = 0.2$. Left: pressure p. Right: velocity $\sqrt{u^2 + v^2}$. Notice the different contour range.

Small perturbation of the 2D equilibrium solution

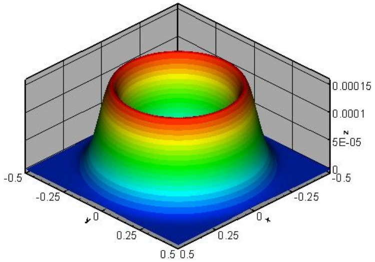
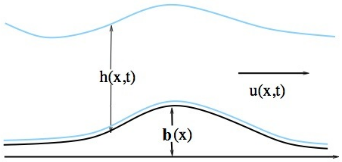


Figure: The 3D views of the velocity $\left(\sqrt{u^2+v^2}\right)$. Left: well-balanced methods; Right: non-well-balanced methods.

SWEs with a non-flat bottom topography

$$\begin{cases} h_t + (hu)_x = 0 \\ (hu)_t + \left(hu^2 + \frac{1}{2}gh^2\right)_x = -ghb_x \end{cases}$$

- h : water height; u : velocity;
  b : bottom topography; g : gravitational constant.



- Source terms (friction and variations of the channel width) can be added.
- Wide applications in coastal ocean, hydraulic engineering and climate.

DG methods for the SWEs

- A vast amount of literatures in Finite Volume methods for the SWEs.
- Initial work by:
  Schwaneberg and Kongeter (2000), Dawson and Proft (CMAME 2002), Giraldo, Hesthaven and Warburton (JCP 2002), Eskilsson and Sherwin (IJNMF 2004).
- Well-balanced DG methods by:
  Xing and Shu (JCP 2006, CiCP 2006), Ern, Piperno and Djadel (IJNMF 2008), Rhebergen, Bokhove and van der Vegt (JCP 2008), Kesserwani and Liang (CF 2010), Xing, Shu and Noelle (JSC 2011), Xing (JCP 2014) Duran and Marche (CF 2014). $\cdots$
- Positivity-preserving DG methods by:
  Bokhove (JSC 2005), Ern, Piperno and Djadel (IJNMF 2008), Bunya, Kubatko, Westerink and Dawson (CMAME 2009), Xing and Zhang (AWR 2010, JSC 2013), Kesserwani and Liang (JSC 2012). $\cdots$

DG methods

Denote the shallow water equations

$$h_t + (hu)_x = 0, \qquad (hu)_t + \left(hu^2 + \frac{1}{2}gh^2\right)_x = -ghb_x,$$

by

$$U_t + f(U)_x = s(h,b),$$

where $U = (h, hu)^T$, $f(U)$ is the flux and $s(h,b)$ is the source term.

DG (semi-discrete) methods

$$\int_{I_j} \partial_t U v dx - \int_{I_j} f(U)\partial_x v dx + \widehat{f}_{j+\frac{1}{2}} v(x^-_{j+\frac{1}{2}}) - \widehat{f}_{j-\frac{1}{2}} v(x^+_{j-\frac{1}{2}}) = \int_{I_j} s(h,b) v dx,$$

where $v(x)$ is a test function belonging to $V_{\Delta x}$,

$$\widehat{f}_{j+\frac{1}{2}} = F(U(x^-_{j+\frac{1}{2}}, t), U(x^+_{j+\frac{1}{2}}, t)),$$

and $F(a_1, a_2)$ is a numerical flux, for example, the simple Lax-Friedrichs flux.

Well-balanced methods

$$\int_{I_j} \partial_t U^n v dx - \int_{I_j} f(U^n)\partial_x v dx + \widehat{f}^l_{j+\frac{1}{2}} v(x^-_{j+\frac{1}{2}}) - \widehat{f}^r_{j-\frac{1}{2}} v(x^+_{j-\frac{1}{2}}) = \int_{I_j} s(h^n, b) v dx.$$

Well-balanced fluxes (Hydrostatic reconstruction)

After computing boundary values $U^{\pm}_{j+\frac{1}{2}}$, we set

$$U^{*,\pm}_{j+\frac{1}{2}} = \left(h^{*,\pm}_{j+\frac{1}{2}},\, h^{*,\pm}_{j+\frac{1}{2}} u^{\pm}_{j+\frac{1}{2}}\right)^T, \quad h^{*,\pm}_{j+\frac{1}{2}} = \max\left(0, h^{\pm}_{j+\frac{1}{2}} + b^{\pm}_{j+\frac{1}{2}} - \max(b^+_{j+\frac{1}{2}}, b^-_{j+\frac{1}{2}})\right).$$

The left and right fluxes $\widehat{f}^l_{j+\frac{1}{2}}$ and $\widehat{f}^r_{j-\frac{1}{2}}$ are given by:

$$\begin{aligned} \widehat{f}^l_{j+\frac{1}{2}} &= F(U^{*,-}_{j+\frac{1}{2}}, U^{*,+}_{j+\frac{1}{2}}) + \begin{pmatrix} 0 \\ \frac{g}{2}(h^-_{j+\frac{1}{2}})^2 - \frac{g}{2}(h^{*,-}_{j+\frac{1}{2}})^2 \end{pmatrix}, \\ \widehat{f}^r_{j-\frac{1}{2}} &= F(U^{*,-}_{j-\frac{1}{2}}, U^{*,+}_{j-\frac{1}{2}}) + \begin{pmatrix} 0 \\ \frac{g}{2}(h^+_{j-\frac{1}{2}})^2 - \frac{g}{2}(h^{*,+}_{j-\frac{1}{2}})^2 \end{pmatrix}. \end{aligned}$$

Note: if $b^+_{i+\frac{1}{2}} = b^-_{i+\frac{1}{2}}$, this is exactly the traditional DG method.

Well-balanced methods for the moving water

- Moving water steady state:

$$m = hu = const, \qquad E = u^2/2 + g(h+b) = const.$$

Challenge: Nonlinearity of E.

- Conservative variables $U = (h, hu)$.
  Equilibrium variables $V = (m, E)$.

- $V = V(U, b)$ is well-defined.
  $U = U(V, b)$ is double-valued.

- To uniquely define U, define the sign function

$$\sigma := \text{sign}(Fr - 1), \qquad Fr := |u|/\sqrt{gh},$$

The flow region is called sonic, sub- or supersonic if $\sigma = 0, -1, 1$.
$U = U(V, b, \sigma)$ is well-defined.

Well-balanced methods for moving water (Xing JCP 2014)

$$\int_{I_j} \partial_t U^n v dx - \int_{I_j} f(U^n) \partial_x v dx + \widehat{f}^l_{j+\frac{1}{2}} v(x^-_{j+\frac{1}{2}}) - \widehat{f}^r_{j-\frac{1}{2}} v(x^+_{j-\frac{1}{2}}) = \int_{I_j} s(h^n, b) v dx,$$

Propoposition: The DG schemes described above maintain smooth moving water equilibrium exactly.

Remarks

- If the bottom is flat, i.e., $b = 0$, our well-balanced DG methods become the traditional DG methods.
- Our well-balanced methods are designed to preserve the moving water equilibrium. When applied to still water steady state, they become existing well-balanced methods presented before.

Positivity-preserving methods

Positivity-preserving methods

- The water height h should be non-negative. Unfortunately, there are negative h in our simulations when high order methods are used.

  Goal: maintain the non-negativity of h.

  Montone scheme (positivity) $\Rightarrow$ first order (Godunov's theorem)
  High order accuracy & positivity $\Rightarrow$ nontrivial.

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  To achieve high order & positivity, we propose:

$$\boxed{h^n(x) \geq 0} \xrightarrow[\textit{step}\,1]{\text{DG update}} \boxed{h^{n+1}(x) \text{ with } \overline{h}^{n+1}(x) \geq 0} \xrightarrow[\textit{step}\,2]{\text{limiter}} \boxed{h^{n+1}(x) \geq 0}$$

Positivity-preserving methods (Step 1)

The cell averages of the water height in the well-balanced DG methods (with a simple Euler forward time discretization):

$$\overline{h}_j^{n+1} = \overline{h}_j^n - \lambda \left[\widehat{F}\left(h^{*,-}_{j+\frac{1}{2}}, u^-_{j+\frac{1}{2}}; h^{*,+}_{j+\frac{1}{2}}, u^+_{j+\frac{1}{2}} \right) - \widehat{F}\left(h^{*,-}_{j-\frac{1}{2}}, u^-_{j-\frac{1}{2}}; h^{*,+}_{j-\frac{1}{2}}, u^+_{j-\frac{1}{2}} \right) \right],$$

where

$$\widehat{F}\left(h^{*,-}_{j+\frac{1}{2}}, u^-_{j+\frac{1}{2}}; h^{*,+}_{j+\frac{1}{2}}, u^+_{j+\frac{1}{2}} \right) = \frac{1}{2}\left(h^{*,-}_{j+\frac{1}{2}} u^-_{j+\frac{1}{2}} + h^{*,+}_{j+\frac{1}{2}} u^+_{j+\frac{1}{2}} - \alpha (h^{*,+}_{j+\frac{1}{2}} - h^{*,-}_{j+\frac{1}{2}}) \right).$$

Proposition

Consider the well-balanced DG methods above. If $h_j^n(x)$ are all non-negative, then $\overline{h}_j^{n+1}$ is also non-negative under the CFL condition ($\widehat{w}_1$ is the Gauss-Lobatto quadrature weight):

$$\lambda\alpha \le \widehat{w}_1.$$

Positivity-preserving limiter (Step 2)

$$\boxed{h^n(x) \geq 0} \quad \xrightarrow[\textit{step 1}]{\text{DG update}} \quad \boxed{h^{n+1}(x) \text{ with } \overline{h}^{n+1}(x) \geq 0} \quad \xrightarrow[\textit{step 2}]{\text{limiter}} \quad \boxed{h^{n+1}(x) \geq 0}$$

Positivity-preserving limiter (Step 2)

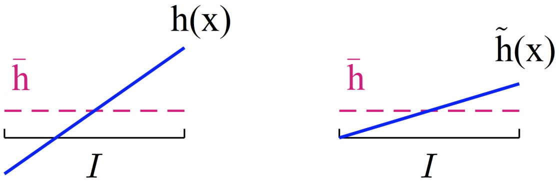
$\boxed{h^n(x) \geq 0}$ $\xrightarrow[\textit{step 1}]{\text{DG update}}$ $\boxed{h^{n+1}(x) \text{ with } \overline{h}^{n+1}(x) \geq 0}$ $\xrightarrow[\textit{step 2}]{\text{limiter}}$ $\boxed{h^{n+1}(x) \geq 0}$

To enforce step 2, we introduce the following limiter on the DG polynomial

$$\widetilde{h}_j^n(x) = \theta\left(h_j^n(x) - \overline{h}_j^n\right) + \overline{h}_j^n, \qquad \theta = \min\left\{1, \frac{\overline{h}_j^n}{\overline{h}_j^n - m_j}\right\},$$

with

$$m_j = \min_{x \in I_j} h_j^n(x).$$



This limiter does not affect the high order accuracy and mass conservation.

Properties of this limiter

- Keeps water height non-negative;
- Preserves the local conservation of h;
- Does not destroy the high order accuracy;
- Only active in the dry or nearly dry region.

Comments

- Works for TVD high order Runge-Kutta and multi-step time discretizations.
- Positivity preserving CFL condition is $\lambda\alpha \leq 1/6$ for $k = 2$, 3. Recall that the CFL condition for linear stability for the DG methods is $\lambda\alpha \leq 1/5$ for $k = 2$.
- Any other positivity-preserving exact or approximate Rieman solver, including Godunov, Boltzmann type and Harten-Lax-Van Leer, can also be used.

Two-dimensional shallow water system

$$\begin{cases} h_t + (hu)_x + (hv)_y = 0 \\ (hu)_t + \left(hu^2 + \frac{1}{2}gh^2\right)_x + (huv)_y = -ghb_x \\ (hv)_t + (huv)_x + \left(hv^2 + \frac{1}{2}gh^2\right)_y = -ghb_y. \end{cases}$$

- Still water at rest steady state:

$$h + b = const, \qquad hu = 0, \qquad hv = 0.$$

- Both well-balanced and positivity-preserving techniques can be extended to 2D.
- Works for both rectangular and triangular meshes.

Summary

Constructed and tested DG methods for the Euler equations.

- Well-balanced approaches to perform efficiently near the steady state:
  1. Isothermal equilibrium state,
  2. Polytropic equilibrium state.

Constructed and tested DG methods for the shallow water equations.

- Well-balanced approaches to perform efficiently near the steady state:
  1. Still water at rest steady state,
  2. Moving water steady state.
- A simple positivity-preserving limiter based on high order DG methods:
  1. Preserve the mass conservation,
  2. Does not affect the high order accuracy for the general solutions.

High order finite difference and finite volume WENO methods have also been designed for these models.

References

- Y. Xing, X. Zhang and C.-W. Shu, **Positivity-preserving high order well-balanced discontinuous Galerkin methods for the shallow water equations**, *Advances in Water Resources*, v33 (2010), pp.1476-1493.
- Y. Xing and X. Zhang, **Positivity-preserving well-balanced discontinuous Galerkin methods for the shallow water equations on unstructured triangular meshes**, *Journal of Scientific Computing*, v57 (2013), pp. 19-41.
- Y. Xing, **Exactly well-balanced discontinuous Galerkin methods for the shallow water equations with moving water equilibrium**, *Journal of Computational Physics*, v257 (2014), pp. 536-553.
- X. Wen, Z. Gao, W.S. Don, Y. Xing and P. Li, **Application of positivity-preserving well-balanced discontinuous Galerkin method to tidal bores problems**, *Computers and Fluids*, v139 (2016), pp. 112-119.
- S. Qian, G. Li, F. Shao and Y. Xing, **Positivity-preserving well-balanced discontinuous Galerkin methods for the shallow water flows in open channels**, *Advances in Water Resources*, v115 (2018), pp. 172-184.
- G. Li and Y. Xing, **Well-balanced discontinuous Galerkin methods for the Euler equations under gravitational fields**, *Journal of Scientific Computing*, v67 (2016), pp. 493-513.
- G. Li and Y. Xing, **Well-balanced discontinuous Galerkin methods with hydrostatic reconstruction for the Euler equations with gravitation**, *Journal of Computational Physics*, v352 (2018), pp. 445-462.

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Thank you!