A New Approach for Designing Moving-Water Equilibria Preserving Schemes for the Shallow Water Equations

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joint work with
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Systems of Balance Laws

\[ U_t + f(U)_x + g(U)_y = S(U) \]

Examples:

- Gas dynamics with pipe-wall friction
- Euler equations with gravity/friction
- shallow water equations with Coriolis forces

Applications:

- astrophysical and atmospheric phenomena in many fields including supernova explosions
- (solar) climate modeling and weather forecasting

\[ U_t + f(U)_x + g(U)_y = \frac{1}{\varepsilon} S(U) \]

Examples:

- low Mach number compressible flows
- low Froude number shallow water flows
- diffusive relaxation in kinetic models

Applications:

- various two-phase flows such as bubbles in water
- unmostly incompressible flows with regions of high compressibility such as underwater explosions
- atmospheric flows
Systems of Balance Laws

\[ U_t + f(U)_x + g(U)_y = S(U) \quad \text{or} \quad U_t + f(U)_x + g(U)_y = \frac{1}{\epsilon} S(U) \]

- **Challenges**: certain structural properties of these hyperbolic problems (conservation or balance law, equilibrium state, positivity, asymptotic regimes, etc.) are essential in many applications;

- **Goal**: to design numerical methods that are not only consistent with the given PDEs, but
  - preserve the structural properties at the discrete level – **well-balanced numerical methods**
  - remain accurate and robust in certain asymptotic regimes of physical interest – **asymptotic preserving numerical methods**

[P. LeFloch; 2014]
Well-Balanced (WB) Methods

\[ U_t + f(U)_x + g(U)_y = S(U) \]

- In many physical applications, solutions of the system are small perturbations of the steady states;
- These perturbations may be smaller than the size of the truncation error on a coarse grid;
- To overcome this difficulty, one can use very fine grid, but in many physically relevant situations, this may be unaffordable;

Goal:
- to design a well-balanced numerical method, that is, the method which is capable of exactly preserving some steady state solutions;
- perturbations of these solutions will be resolved on a coarse grid in a non-oscillatory way.
Finite-Volume Methods – 1-D

\[ U_t + f(U)_x = S \]

- \( \overline{U}_k^n \approx \frac{1}{\Delta y} \int_{C_k} U(y, t^n) \, dy \): cell averages over \( C_j := (x_{j-\frac{1}{2}}, x_{j+\frac{1}{2}}) \)

- Semi-discrete FV method:
  \[
  \frac{d}{dt} \overline{U}_j(t) = -\frac{\mathcal{F}_{j+\frac{1}{2}}(t) - \mathcal{F}_{j-\frac{1}{2}}(t)}{\Delta x} + \overline{S}_j
  \]
  \( \mathcal{F}_{j+\frac{1}{2}}(t) \): numerical fluxes
  \( \overline{S}_j \): quadrature approximating the corresponding source terms

- Central-Upwind (CU) Scheme:
\[ \{ \overline{U}_j(t) \} \rightarrow \tilde{U}(\cdot, t) \rightarrow \{ U_{j}^{E,W}(t) \} \rightarrow \{ \mathcal{F}_{j+\frac{1}{2}}(t) \} \rightarrow \{ \overline{U}_j(t + \Delta t) \} \]

(Discontinuous) piecewise-linear reconstruction:

\[ \tilde{U}(y, t) := \overline{U}_j(t) + (U_x)_j(x - x_j), \quad x \in C_j \]

It is conservative, second-order accurate, and non-oscillatory provided the slopes, \( \{(U_y)_k\} \), are computed by a nonlinear limiter.

Example — Generalized Minmod Limiter

\[
(U_y)_j = \text{minmod} \left( \frac{\theta \overline{U}_j - \overline{U}_{j-1}}{\Delta x}, \frac{\overline{U}_{j+1} - \overline{U}_{j-1}}{2\Delta x}, \frac{\theta \overline{U}_{j+1} - \overline{U}_j}{\Delta x} \right)
\]

where

\[
\text{minmod}(z_1, z_2, ...) := \begin{cases} 
\min_j \{z_j\}, & \text{if } z_j > 0 \ \forall j, \\
\max_j \{z_j\}, & \text{if } z_j < 0 \ \forall j, \\
0, & \text{otherwise,}
\end{cases}
\]

and \( \theta \in [1, 2] \) is a constant.
\[
\{\overline{U}_j(t)\} \rightarrow \widetilde{U}(\cdot, t) \rightarrow \{U_{j}^{E,W}(t)\} \rightarrow \{\mathcal{F}_{j+\frac{1}{2}}(t)\} \rightarrow \{\overline{U}_j(t + \Delta t)\}
\]

\(U_j^E\) and \(U_j^W\) are the point values at \(x_{j+\frac{1}{2}}\) and \(x_{j-\frac{1}{2}}\):

\[
\widetilde{U}(y, t) = \overline{U}_j + (U_x)_j(x - x_j), \quad x \in C_j
\]

\[
U_j^E := \overline{U}_j + \frac{\Delta x}{2}(U_x)_j
\]

\[
U_j^W := \overline{U}_j - \frac{\Delta x}{2}(U_x)_j
\]
\[
\{ \overline{U}_j(t) \} \rightarrow \tilde{\mathbf{U}}(\cdot, t) \rightarrow \left\{ \mathbf{U}^{E,W}_j(t) \right\} \rightarrow \left\{ \mathbf{F}_{j+\frac{1}{2}}(t) \right\} \rightarrow \{ \overline{U}_j(t + \Delta t) \}
\]

\[
\frac{d}{dt} \overline{U}_j = -\frac{\mathbf{F}_{j+\frac{1}{2}} - \mathbf{J}_{j-\frac{1}{2}}}{\Delta x} + \mathbf{S}_j
\]

where

\[
\mathbf{F}_{j+\frac{1}{2}} = \frac{\alpha_{j+\frac{1}{2}} f(\mathbf{U}_j^E) - \alpha_{j+\frac{1}{2}} f(\mathbf{U}_{j+1}^W)}{a_{j+\frac{1}{2}} - a_{j+\frac{1}{2}}} + \alpha_{j+\frac{1}{2}} (\mathbf{U}_{j+1}^W - \mathbf{U}_j^W)
\]

\[
\alpha_{j+\frac{1}{2}} = \frac{a_{j+\frac{1}{2}} a_{j+\frac{1}{2}}}{a_{j+\frac{1}{2}} - a_{j+\frac{1}{2}}}
\]

\[
a_{j+\frac{1}{2}} = \max \left\{ \lambda(\mathbf{U}_j^E), \lambda(\mathbf{U}_{j+1}^W), 0 \right\}, \quad a_{j+\frac{1}{2}} = \min \left\{ \lambda(\mathbf{U}_j^E), \lambda(\mathbf{U}_{j+1}^W), 0 \right\}
\]

2-D extension is dimension-by-dimension
Non Well-Balanced Property – Example

\[
\begin{align*}
  h_t + q_x &= 0, \\
  q_t + f_2(h, q)_x &= -s(h, q)
\end{align*}
\]

For steady-state solution: \( q = \text{Const} \) and \( h = h(x) \)

Implementing the CU scheme results in

\[
\frac{d \bar{h}_j}{dt} = -\frac{1}{\Delta x} \left[ \frac{a_{j+\frac{1}{2}}^+ q_{j+\frac{1}{2}} - a_{j+\frac{1}{2}}^- q_{j+\frac{1}{2}+1}}{a_{j+\frac{1}{2}}^+ - a_{j+\frac{1}{2}}^-} + \alpha_{j+\frac{1}{2}} (h_{j+1}^W - h_j^E) ight]
\]

- The steady state would not be preserved at the discrete level;
- This would also be true for the first-order version of the scheme;
- For smooth solutions, the balance error is expected to be of order \((\Delta x)^2\), but a coarse grid solution may contain large spurious waves.
Well-Balanced Methods
1-D $2 \times 2$ Systems of Balance Laws

\[
\begin{aligned}
ht + f_1(h, q)_x &= 0, \\
q_t + f_2(h, q)_x &= -s(h, q),
\end{aligned}
\]

**Steady state solution:**

\[f_1(h, q)_x \equiv 0, \quad f_2(h, q)_x + s(h, q) \equiv 0\]

or

\[K := f_1(h, q) \equiv \text{Const},\]

\[L := f_2(h, q) + \int s(h, q)d\xi \equiv \text{Const} \quad \forall x, t\]

Numerical Challenges: to exactly balance the flux and source terms, i.e., to exactly preserve the steady states.

**How to design a well-balanced scheme?**
Well-Balanced Scheme

\[
\begin{cases}
    h_t + f_1(h, q)_x = 0, \\
    q_t + f_2(h, q)_x = -s(h, q)
\end{cases}
\]

- Incorporate the source term into the flux:

\[
\begin{cases}
    h_t + f_1(h, q)_x = 0, \\
    q_t + (f_2(h, q)_x + R)_x = 0, \\
    R := \int_x s(h, q) d\xi
\end{cases}
\]

- Rewrite

\[
\begin{cases}
    h_t + K_x = 0, \\
    q_t + L_x = 0
\end{cases}
\]

where

\[K := f_1(h, q), \quad L := f_2(h, q)_x + R\]

- Define

  conservative variables \( \mathbf{U} = (h, q)^T \)

  equilibrium variables \( \mathbf{W} := (K, L)^T \)
Well-Balanced Scheme

\[ U_t + f(U)_x = 0 \]

\[ U = \begin{pmatrix} h \\ q \end{pmatrix}, \quad f(U) = W := \begin{pmatrix} K \\ L \end{pmatrix} \]

Semi-discrete FV method:

\[ \frac{d}{dt} \overline{U}_j(t) = -\frac{\mathcal{F}_{j+\frac{1}{2}}(t) - \mathcal{F}_{j-\frac{1}{2}}(t)}{\Delta x} \]

**Two major modifications:**

- **Well-balanced reconstruction** – performed on the equilibrium rather than conservative variables:

  \[ \{ \overline{U}_j(t) \} \rightarrow \tilde{U}(\cdot , t) \rightarrow \left\{ W_j^{E,W}(t) \right\} \rightarrow \left\{ U_j^{E,W}(t) \right\} \rightarrow \left\{ \mathcal{F}_{j}^{+\frac{1}{2}}(t) \right\} \rightarrow \{ \overline{U}_j(t+\Delta t) \} \]

- **Well-balanced evolution**
Well-Balanced Reconstruction

**Given:** $\bar{U}_j(t) = (\bar{h}_j, \bar{q}_j)^T$ – cell averages

**Need:** $W_{j}^{E,W} = (K_j^{E,W}, L_j^{E,W})^T$ – point values, where

$$K := f_1(h, q), \quad L := f_2(h, q)_x + R, \quad R := \int_{x_j}^{x} s(h, q) d\xi$$

- Compute $R_j = \int_{x_j}^{x} s(h, q) d\xi$ by the midpoint quadrature rule and using the following recursive relation:

  $$R_{1/2} \equiv 0, \quad R_j = \frac{1}{2}(R_{j-\frac{1}{2}} + R_{j+\frac{1}{2}}),$$
  $$R_{j+\frac{1}{2}} = R(x_{j+\frac{1}{2}}) = R_{j-\frac{1}{2}} + \Delta x s(x_j, \bar{h}_j, \bar{q}_j)$$

- Compute the point values of $K$ and $L$ at $x_j$ from the cell averages, $\bar{h}_j$ and $\bar{q}_j$:

  $$K_j = f_1(\bar{h}_j, \bar{q}_j), \quad L_j = f_2(\bar{h}_j, \bar{q}_j) + R_j$$
Well-Balanced Reconstruction

- Apply the minmod reconstruction procedure to $\{K_j, L_j\}$ and obtain the point values at the cell interfaces:

$$K_j^E = K_j + \frac{\Delta x}{2}(K_x)_j, \quad L_j^E = L_j + \frac{\Delta x}{2}(L_x)_j,$$

$$K_j^W = K_j - \frac{\Delta x}{2}(K_x)_j, \quad L_j^W = L_j - \frac{\Delta x}{2}(L_x)_j,$$

- Finally, equipped with the values of $K_j^{E,W}, L_j^{E,W}$ and $R_{j\pm\frac{1}{2}}$, solve

$$K_j^E = f_1(h_j^E, q_j^E), \quad L_j^E = f_2(h_j^E, q_j^E) + R_{j+\frac{1}{2}},$$

$$K_j^W = f_1(h_j^W, q_j^W), \quad L_j^W = f_2(h_j^W, q_j^W) + R_{j-\frac{1}{2}},$$

for $U_j^{E,W} = (h_j^{E,W}, q_j^{E,W})^T$. 
Well-Balanced Evolution

\[ \frac{d}{dt} U_j = - \frac{\mathcal{F}_{j+\frac{1}{2}} - \mathcal{F}_{j-\frac{1}{2}}}{\Delta x} \]

where

\[ \mathcal{F}^{(1)}_{j+\frac{1}{2}} = \frac{a^+_{j+\frac{1}{2}} K^E_j - a^-_{j+\frac{1}{2}} K^W_{j+1}}{a^+_{j+\frac{1}{2}} - a^-_{j+\frac{1}{2}}} \]

\[ + \alpha_{j+\frac{1}{2}} (h^W_{j+1} - h^E_{j}) \mathcal{H}\left(\frac{|K_{j+1} - K_j|}{\Delta x} \cdot \frac{|\Omega|}{\max_j K_j, K_{j+1}}\right) \]

\[ \mathcal{F}^{(2)}_{j+\frac{1}{2}} = \frac{a^+_{j+\frac{1}{2}} L^E_j - a^-_{j+\frac{1}{2}} L^W_{j+1}}{a^+_{j+\frac{1}{2}} - a^-_{j+\frac{1}{2}}} \]

\[ + \alpha_{j+\frac{1}{2}} (q^W_{j+1} - q^E_{j}) \mathcal{H}\left(\frac{|L_{j+1} - L_j|}{\Delta x} \cdot \frac{|\Omega|}{\max_j \{L_j, L_{j+1}\}}\right) \]
Proof of the Well-Balanced Property

**Theorem.** The central-upwind semi-discrete schemes coupled with the well-balanced reconstruction and evolution is well-balanced in the sense that it preserves the corresponding steady states exactly.
1-D Saint-Venant System of Shallow Water with Friction

\[
\begin{align*}
    & h_t + q_x = 0 \\
    & q_t + \left( hu^2 + \frac{g}{2} h^2 \right)_x = -ghB_x - g\frac{n^2}{h^{7/3}} |q| q
\end{align*}
\]

- \( h \) – water depth
- \( u \) – velocity
- \( q := hu \) – discharge
- \( B(x) \) – bottom elevation
- \( g \) – the constant gravitational acceleration
- \( n \) – Manning friction coefficient.

\[ w = B + h \]

\[ h(x,t) \]
Shallow Water Equations

\[
\begin{align*}
    h_t + q_x &= 0 \\
    q_t + \left( hu^2 + \frac{g}{2} h^2 \right)_x &= -ghB_x - g\frac{n^2}{h^{7/3}}|q|q
\end{align*}
\]

- **Well-balanced scheme** should exactly balance the flux and source terms so that the steady states are preserved:
  
  - Moving Steady-state solutions (**no friction** \( n \equiv 0 \)):
    
    \[
    q = \text{Const}, \quad \frac{u^2}{2} + g(h + B) = \text{Const}
    \]
  
  - Stationary steady-state solutions (**lake at rest**):
    
    \[
    u = 0, \quad h + B = \text{Const}
    \]
Well-Balanced Methods – Some References

- Shallow water models (preserving “lake at rest” steady states):
  - LeVeque (1998) – incorporating the source term into the Riemann solver
  - Jin (2001) – well-balanced source term averaging
  - Perthame, Simeoni (2001) – kinetic scheme
  - Gallouët, Hérard, Seguin (2003) – Roe-type scheme
  - Russo (2005) staggered central scheme
  - Xing, Shu (2005, 2006) – WENO schemes
  - Noelle, Pankratz, Puppo, Natvig (2006) – high-order schemes
  - Lukácová-Medvidová, Noelle, Kraft (2007) FVEG scheme
  - Berthon, Marche (2008) – relaxation schemes
  - Fjordholm, Mishra, Tadmor (2008, 2011) energy stable schemes
  - Abgrall, Audusse, Bristeau, Castro, Chertock, Dawson, Donat, Epshteyn, George, Karni, Klingenberg, Mohammadian, Parés, Ricchiuto, ...
Well-Balanced Methods – Some References

- **Shallow water models (preserving moving steady states):**
  - Russo, Khe (2009, 2010) – staggered central schemes
  - Xing (2014) – discontinuous Galerkin method

- **Shallow water models (positivity preserving schemes):**
  - Perthame, Simeoni (2001) – kinetic scheme
  - Kurgnov, Petrova (2007) – central-upwind scheme with continuous piecewise linear bottom reconstruction
  - Berthon, Marche (2008) – relaxation schemes
  - Bollermann, Noelle, Lukáčová-Medvid’ová (2011) – special time-quadrature for the fluxes
  - Bollermann, Chen, Kurganov, Noelle (2013): well-balanced reconstruction of wet/dry fronts
Moving Steady States with Friction

\[
\begin{align*}
\partial_t h + q_x &= 0, \\
\partial_t q + \left( hu^2 + \frac{g}{2} h^2 \right)_x &= -ghB_x - g\frac{n^2}{h^{7/3}}|q|q
\end{align*}
\]

We incorporate the source term in the discharge equation into its flux term:

\[
\begin{align*}
\partial_t h + q_x &= 0, \\
\partial_t q + \left( q^2 + \frac{g}{2} h^2 + R \right)_x &= 0
\end{align*}
\]

General (moving-water) steady state can be expressed in terms of $K$ and $L$:

\[
q \equiv \text{Const}, \quad K \equiv \text{Const}
\]

where

\[
K := \frac{q^2}{h} + \frac{g}{2} h^2 + R
\]

\[
R(x, t) := g \int x \left[ h(\xi, t)B_x(\xi) + \frac{n^2}{h^{7/3}(\xi)}|q(\xi)||q(\xi)| \right] d\xi
\]
Well-Balanced Algorithm

**Given:** $\overline{U}_j(t) = (\overline{h}_j, \overline{q}_j)^T$ – cell averages

- Compute equilibrium variables $(\overline{q}_j, K_j)^T$ at $x_j$ from the above cell averages:
  
  $$\overline{q}_j, \quad K_j = \frac{\overline{q}_j^2}{\overline{h}_j} + \frac{g}{2} \overline{h}_j^2 + \frac{R_{j+\frac{1}{2}} + R_{j-\frac{1}{2}}}{2},$$

  where

  $$R(x_{j+\frac{1}{2}}, t) \approx R_{j+\frac{1}{2}} := g \sum_{m=j_{\ell}}^{j} \left\{ \overline{h}_m (B_{m+\frac{1}{2}} - B_{m-\frac{1}{2}}) + \frac{n^2}{\overline{h}_m^{7/3}}|\overline{q}_m|\overline{q}_m \Delta x \right\}$$

  $$= R_{j-\frac{1}{2}} + \frac{g}{2} \left[ \overline{h}_j (B_{j+\frac{1}{2}} - B_{j-\frac{1}{2}}) + \frac{n^2}{\overline{h}_j^{7/3}}|\overline{q}_j|\overline{q}_j \Delta x \right]$$

- Apply the minmod reconstruction procedure to $\{K_j, L_j\}$ and obtain the point values at the cell interfaces:

  $$q_{j}^{E,W} = q_j \pm \frac{\Delta x}{2} (q_x)_j, \quad K_{j}^{E,W} = K_j \pm \frac{\Delta x}{2} (K_x)_j$$
Well-Balanced Algorithm

• Compute point values \( h_j^{E,W} \) by solving the nonlinear algebraic equations

\[
\varphi(h) := \frac{(q_j^{E,W})^2}{h} + \frac{g}{2}h^2 + R_{j \pm \frac{1}{2}} - K_j^{E,W} = 0,
\]

which does not have any positive solutions unless

\[
(q_j^{E,W})^4 \leq \frac{8(K_j^{E,W} - R_{j \pm \frac{1}{2}})^3}{27g}
\]

Consider \( h_j^E \)
Well-Balanced Algorithm

• If the inequality is not satisfied, we reconstruct \( w = \bar{h} + B \) and set

\[
\bar{h}_j^E = w_j^E - B_{j+\frac{1}{2}}
\]

\( (*) \)

• If the inequality is satisfied, then

– If \( q_j^E = 0 \), then

\[
\bar{h}_j^E = \sqrt{\frac{2(K_j^E - R_{j+\frac{1}{2}})}{g}}
\]

– If \( q_j^E \neq 0 \), then

\[
\bar{h}_j^E = 2\sqrt{P}\cos\left(\frac{1}{3}[\Theta + 2\pi k]\right), \quad k = 0, 1, 2,
\]

where

\[
P := \frac{2(K_j^E - R_{j+\frac{1}{2}})}{3g} \quad \text{and} \quad \Theta := \arccos\left(-\frac{(q_j^E)^2}{gP^{3/2}}\right)
\]

Only two roots are positive (subsonic and supersonic cases). We single out the physically relevant solution by choosing a root that is closer to the corresponding value of \( \bar{h}_j^E \) given in \( (*) \).
Well-Balanced Algorithm

• Update the cell averages in time:

\[
\frac{d}{dt} \bar{h}_j = \frac{a^+_{j+\frac{1}{2}} q^E_j - a^-_{j+\frac{1}{2}} q^W_{j+1}}{a^+_{j+\frac{1}{2}} - a^-_{j+\frac{1}{2}}} + \frac{a^+_{j+\frac{1}{2}} a^-_{j+\frac{1}{2}}}{a^+_{j+\frac{1}{2}} - a^-_{j+\frac{1}{2}}} (h^W_{j+1} - h^E_j)
\]

\[
\frac{d}{dt} \bar{q}_j = \frac{a^+_{j+\frac{1}{2}} K^E_j - a^-_{j+\frac{1}{2}} K^W_{j+1}}{a^+_{j+\frac{1}{2}} - a^-_{j+\frac{1}{2}}} + \frac{a^+_{j+\frac{1}{2}} a^-_{j+\frac{1}{2}}}{a^+_{j+\frac{1}{2}} - a^-_{j+\frac{1}{2}}} (q^W_{j+1} - q^E_j)
\]

with

\[
a^+_{j+\frac{1}{2}} = \max \left\{ u^W_{j+1} + \sqrt{gh^W_{j+1}}, u^E_{j} + \sqrt{gh^E_{j}}, 0 \right\}
\]

\[
a^-_{j+\frac{1}{2}} = \min \left\{ u^W_{j+1} + \sqrt{gh^W_{j+1}}, u^E_{j} + \sqrt{gh^E_{j}}, 0 \right\}
\]
Numerical Tests
Example 1 – Accuracy Test, No Friction

- Initial data and the bottom topography: function are

\[ h(x, 0) = 5 + e^{\cos(2\pi x)}, \quad q(x, 0) = \sin(\cos(2\pi x)), \quad B(x) = \sin^2(\pi x) \]

- 1-periodic boundary conditions are imposed on \([0, 1]\)
- Reference solution is computed on a very fine mesh with 51200 uniform grid cells, time is \(t = 0.1\).

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<th>Number of grid cells</th>
<th>(h) (L^1)-error</th>
<th>Rate</th>
<th>(q) (L^1)-error</th>
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<td>2.01</td>
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</tbody>
</table>
Example 2 – Convergence to Steady States, no Friction

- Three sets of initial conditions:
  - Supercritical flow with
    \[ h(x, 0) = 2 - B(x), \quad q(x, 0) \equiv 0, \]
    \[ h(0, t) = 2, \quad q(0, t) = 24; \]
  - Subcritical flow with
    \[ h(x, 0) = 2 - B(x), \quad q(x, 0) \equiv 0, \]
    \[ q(0, t) = 4.42, \quad h(25, t) = 2; \]
  - Transcritical flow without a shock with
    \[ h(x, 0) = 0.66 - B(x), \quad q(x, 0) \equiv 0, \]
    \[ q(0, t) = 1.53, \quad h(25, t) = 0.66. \]

- **Continuous** bottom topography:

\[
B(x) = \begin{cases} 
0.2 - 0.05(x - 10)^2, & \text{if } 8 \leq x \leq 12, \\
0, & \text{otherwise}.
\end{cases}
\]
Example 3 – Convergence to Steady States, with Friction \((n = 0.5)\)

- The same three sets of initial conditions as in Example 2:
  - Supercritical flow with
    \[
    h(x, 0) = 2 - B(x), \quad q(x, 0) \equiv 0, \\
    h(0, t) = 2, \quad q(0, t) = 24;
    \]
  - Subcritical flow with
    \[
    h(x, 0) = 2 - B(x), \quad q(x, 0) \equiv 0, \\
    q(0, t) = 4.42, \quad h(25, t) = 2;
    \]
  - Transcritical flow without a shock with
    \[
    h(x, 0) = 0.66 - B(x), \quad q(x, 0) \equiv 0, \\
    q(0, t) = 1.53, \quad h(25, t) = 0.66.
    \]

- **Continuous** bottom topography:
  \[
  B(x) = \begin{cases} 
  0.2 - 0.05(x - 10)^2, & \text{if } 8 \leq x \leq 12, \\
  0, & \text{otherwise}.
  \end{cases}
  \]
Example 4 – Convergence to Steady States, with Friction \((n = 0.5)\)

• The same three sets of initial conditions as in Example 2:
  – Supercritical flow with
    
    \[ h(x, 0) = 2 - B(x), \quad q(x, 0) \equiv 0, \]
    
    \[ h(0, t) = 2, \quad q(0, t) = 24; \]
  – Subcritical flow with
    
    \[ h(x, 0) = 2 - B(x), \quad q(x, 0) \equiv 0, \]
    
    \[ q(0, t) = 4.42, \quad h(25, t) = 2; \]
  – Transcritical flow without a shock with
    
    \[ h(x, 0) = 0.66 - B(x), \quad q(x, 0) \equiv 0, \]
    
    \[ q(0, t) = 1.53, \quad h(25, t) = 0.66. \]

• **Discontinuous** bottom topography:

  \[
  B(x) = \begin{cases} 
  0.2, & \text{if } 8 \leq x \leq 12, \\
  0, & \text{otherwise}. 
  \end{cases}
  \]
Example 5 – Small Perturbations of Moving-Water Equilibria, with Friction ($n = 0.5$)

- Two sets of initial conditions:
  - Supercritical flow with
    
    \[ q(x, 0) \equiv 24, \quad K(x, 0) \equiv 307.624, \]
  - Subcritical flow with
    
    \[ q(x, 0) \equiv 4.42, \quad K(x, 0) \equiv 31.7705, \]

- **Discontinuous** bottom topography:

  \[ B(x) = \begin{cases} 
  0.2, & \text{if } 8 \leq x \leq 12, \\
  0, & \text{otherwise.} 
\end{cases} \]

We add 0.001 for $x \in [4.5, 5.5]$ to the corresponding water depth. We compute the solutions until the final time $t = 1$ using either 100 or 1000 uniform grid cells.
Shallow Water System with Coriolis Force

\[
\begin{aligned}
&h_t + (hu)_x + (hv)_y = 0 \\
&(hu)_t + \left( hu^2 + \frac{g}{2}h^2 \right)_x + (huv)_y = -ghB_x + fhv \\
&(hv)_t + (huv)_x + \left( hv^2 + \frac{g}{2}h^2 \right)_x = -ghB_y - fhu
\end{aligned}
\]

- \( h \): water height
- \( u, v \): fluid velocity
- \( B \): bottom topography
- \( g \): gravitational constant
- \( f \): Coriolis parameter; \( f \equiv 0 \) \( \implies \) Saint Venant system of shallow water.
Steady States

\[
\begin{cases}
    h_t + (hu)_x + (hv)_y = 0 \\
    (hu)_t + \left( hu^2 + \frac{g}{2} h^2 \right)_x + (huv)_y = -ghB_x + fhv \\
    (hv)_t + (huv)_x + \left( hv^2 + \frac{g}{2} h^2 \right)_y = -ghB_y - fhu
\end{cases}
\]

- "Lake at rest": \( u \equiv 0, \ v \equiv 0, \ h + B \equiv \text{Const} \)
- Geostrophic equilibria ("jets in the rotational frame") are both stationary and constant along the streamlines:
  \[
  \begin{align*}
    u &\equiv 0, \ v_y \equiv 0, \ h_y \equiv 0, \ B_y \equiv 0, \ K \equiv \text{Const} \\
    v &\equiv 0, \ u_x \equiv 0, \ h_x \equiv 0, \ B_x \equiv 0, \ L \equiv \text{Const}
  \end{align*}
  \]

Here,  

\[
K := g(h + B - V) \quad \text{and} \quad L := g(h + B + U)
\]

are the potential energies defined through the primitives of the Coriolis force \((U, V)^T\):

\[
V_x := \frac{f}{g}v \quad \text{and} \quad U_y := \frac{f}{g}u
\]
2-D Well-Balanced Scheme

- Define

  conservative variables: \( \mathbf{U} := (h, hu, hv)^T \)

  equilibrium variables: \( \mathbf{W} := (u, v, K, L)^T \)

  fluxes in the \( x \)- and \( y \)-directions: \( f(\mathbf{U}, B) \) and \( g(\mathbf{U}, B) \)

- Assume that at time \( t \) the cell averages are available

  \[
  \bar{\mathbf{U}}_{j,k}(t) := \frac{1}{\Delta x \Delta y} \int \int_{C_{j,k}} \mathbf{U}(x, y, t) \, dx \, dy,
  \]

- Solve by the well-balanced scheme

  \[
  \{\bar{\mathbf{U}}_{j,k}(t)\} \rightarrow \tilde{\mathbf{U}}(\cdot, t) \rightarrow \left\{ \mathbf{W}_{j,k}^{E,W,N,S}(t) \right\} \rightarrow \left\{ \mathbf{U}_{j,k}^{E,W,N,S}(t) \right\} \rightarrow \left\{ \mathbf{F}_{j+\frac{1}{2},k}(t), \mathbf{G}_{j,k+\frac{1}{2}}(t) \right\} \rightarrow \{\bar{\mathbf{U}}_{j,k}(t + \Delta t)\} \]
Example — 2-D Stationary Vortex


\[ h(r, 0) = 1 + \varepsilon^2 \begin{cases} 
\frac{5}{2} (1 + 5\varepsilon^2) r^2 \\
\frac{1}{10} (1 + 5\varepsilon^2) + 2r - \frac{1}{2} - \frac{5}{2} r^2 + \varepsilon^2 (4 \ln(5r) + \frac{7}{2} - 20r + \frac{25}{2} r^2) \\
\frac{1}{5} (1 - 10\varepsilon + 4\varepsilon^2 \ln 2), 
\end{cases} \]

\[ u(x, y, 0) = -\varepsilon y \Upsilon(r), \quad v(x, y, 0) = \varepsilon x \Upsilon(r), \quad \Upsilon(r) := \begin{cases} 
5, & r < \frac{1}{5} \\
2 - \frac{1}{r} - 5, & \frac{1}{5} \leq r < \frac{2}{5} \\
0, & r \geq \frac{2}{5},
\end{cases} \]

Domain: \([-1, 1] \times [-1, 1], \quad r := \sqrt{x^2 + y^2}\]

Boundary conditions: a zero-order extrapolation in both \(x\)- and \(y\)-directions

Parameters: \(B \equiv 0, \quad f = 1/\varepsilon\) and \(g = 1/\varepsilon^2\) with \(\varepsilon = 0.05\)
LIFE IS LIKE MATH
IF IT GOES TOO EASY SOMETHING IS WRONG
Asymptotic Perserving Methods
Explicit Discretization

Eigenvalues of the flux Jacobian:

\[
\begin{align*}
\{ u \pm \frac{1}{\varepsilon} \sqrt{h}, u \} \quad \text{and} \quad \{ v \pm \frac{1}{\varepsilon} \sqrt{h}, v \}
\end{align*}
\]

This leads to the CFL condition

\[
\Delta t_{\text{expl}} \leq \nu \cdot \min \left( \frac{\Delta x}{\max \left\{ |u| + \frac{1}{\varepsilon} \sqrt{h} \right\}}, \frac{\Delta y}{\max \left\{ |v| + \frac{1}{\varepsilon} \sqrt{h} \right\}} \right) = O(\varepsilon \Delta_{\text{min}}).
\]

where \( \Delta_{\text{min}} := \min(\Delta x, \Delta y) \)

- \( 0 < \nu \leq 1 \) is the CFL number
- Numerical diffusion: \( O(\lambda_{\text{max}} \Delta x) = O(\varepsilon^{-1} \Delta x) \).
- We must choose \( \Delta x \approx \varepsilon \) to control numerical diffusion and the stability condition becomes

\[
\Delta t = O(\varepsilon^2)
\]
Low Froude Number Flows

Low Froude number regime ($0 < \varepsilon \ll 1$) $\implies$ very large propagation speeds

Explicit methods:
- very restrictive time and space discretization steps, typically proportional to $\varepsilon$ due to the CFL condition;
- too computationally expensive and typically impractical.

Implicit schemes:
- uniformly stable for $0 < \varepsilon < 1$;
- may be inconsistent with the limit problem;
- may provide a wrong solution in the zero Froude number limit.

Goal: to design robust numerical algorithms, whose accuracy and efficiency is independent of $\varepsilon$
Some References

• Harlow, Welch; 1965
• Chorin; 1967
• Harlow, Amsden; 1971
• Klainerman, Majda; 1981
• Turkel; 1987
• Abarbanel, Duth, Gottlieb; 1989
• Gustafsson, Stoor; 1991
• Klein; 1995
• Colella, Pao; 1999
• Guillard, Viozat; 1999
• Guillard, Murrone; 2004
• Kadioglu, Sussman, Osher, Wright, Kang; 2005
Asymptotic-Preserving (AP) Methods

Introduced in [Klar; 1998, Jin; 1999], see also [Jin, Levermore; 1991], [Golse, Jin, Levermore; 1999].

Idea:
• asymptotic passage from one model to another should be preserved at the discrete level;
• for a fixed mesh size and time step, AP method should automatically transform into a stable discretization of the limiting model as \( \varepsilon \to 0 \).

Figure 7: Properties of AP-schemes

0.4 Outline

The present work is a review of several Asymptotic-Preserving schemes, constructed in the kinetic and fluid framework. Inevitably, the choice of the model problems is related with the author's knowledge and with the concept of providing the reader with the most important features of AP-schemes. These schemes can be designed for several other singularly perturbed problems, that admit asymptotic behaviours/regimes.

An overview of the subject of this manuscript is:
– Chapter 1 deals with the Boltzmann equation in the drift-diffusion limit
– Chapter 2 discusses the Vlasov-Poisson system in the quasi-neutral limit
– Chapter 3 treats the subject of the Vlasov equation in the high-field limit and considering variable Larmor radii
– Chapter 4 introduces an Asymptotic-Preserving scheme for a highly elliptic potential equation
– Chapter 5 deals finally with a highly anisotropic, nonlinear, degenerate parabolic temperature equation.
AP Methods – References

[Degond, Jin, Liu; 2007]
[Degond, Hua, Navoret; 2011]
[Degond and M. Tang; 2011]
[Berthon, Turpault; 2011]
[Cordier, Degond, Kumbaro; 2012]
[Haack, Jin, Liu; 2012]
[Arun, Noelle, Lukáčová-Medvid’ová, Munz; 2014]
[Miczek, Roepke, Edelmann; 2015]
[Bispen, Lukáčová-Medvid’ová, Yelash; 2017]
[Feireisl, Klingenberg, Markfelder; preprint 2017]

Though the existing AP schemes work perfectly well for many simpler models, their applicability to more complicated systems is rather limited: They work very well for large ($\varepsilon \sim 1$) and intermediate ($\varepsilon \sim 10^{-1}$) values of $\varepsilon$, but may become inefficient for smaller $\varepsilon$ numbers.
Theorem. A new hyperbolic flux splitting method coupled with the described fully discrete scheme, which is asymptotic preserving in the sense that it provides a consistent and stable discretization of the limiting system as the Froude number $\varepsilon \to 0$.

Joint work with Alexander Kurganov and Xin Liu
Example — 2-D Stationary Vortex

\[
\begin{align*}
    h(r, 0) &= 1 + \varepsilon^2 \begin{cases} 
        \frac{5}{2} (1 + 5\varepsilon^2) r^2 \\
        \frac{1}{10} (1 + 5\varepsilon^2) + 2r - \frac{1}{2} - \frac{5}{2} r^2 + \varepsilon^2 (4 \ln(5r) + \frac{7}{2} - 20r + \frac{25}{2} r^2) \\
        \frac{1}{5} (1 - 10\varepsilon + 4\varepsilon^2 \ln 2),
    \end{cases} \\
    u(x, y, 0) &= -\varepsilon y \Upsilon(r), \quad v(x, y, 0) = \varepsilon x \Upsilon(r), \quad \Upsilon(r) := \begin{cases} 
        5, & r < \frac{1}{5} \\
        \frac{2}{r} - 5, & \frac{1}{5} \leq r < \frac{2}{5} \\
        0, & r \geq \frac{2}{5},
    \end{cases}
\end{align*}
\]

Domain: \([-1, 1] \times [-1, 1], \quad r := \sqrt{x^2 + y^2}

Boundary conditions: a zero-order extrapolation in both \(x\)- and \(y\)-directions
Comparison of non-AP and AP methods, $\varepsilon = 1$
Comparison of non-AP and AP methods, $\varepsilon = 0.1$
Comparison of non-AP and AP methods, $\varepsilon = 0.01$
Comparison of non-AP and AP methods, CPU times

<table>
<thead>
<tr>
<th>Grid</th>
<th>$\varepsilon = 1$</th>
<th>$\varepsilon = 0.1$</th>
<th>$\varepsilon = 0.01$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>AP</td>
<td>Explicit</td>
<td>AP</td>
</tr>
<tr>
<td>40 $\times$ 40</td>
<td>0.18 s</td>
<td>0.16 s</td>
<td>0.06 s</td>
</tr>
<tr>
<td>80 $\times$ 80</td>
<td>1.57 s</td>
<td>1.32 s</td>
<td>0.29 s</td>
</tr>
<tr>
<td>200 $\times$ 200</td>
<td>24.11 s</td>
<td>21.36 s</td>
<td>5.36 s</td>
</tr>
</tbody>
</table>
Smaller values: $\varepsilon = 10^{-3}$ and $\varepsilon = 10^{-4}$

Smaller times: $200 \times 200$, larger times: $500 \times 500$
THANK YOU!