Well-balanced schemes for gas-flow in pipeline networks

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Solutions to the classical balance law

Classical solutions: $U \in C^1(\mathbb{R} \times [0,T))$ solves

$$\begin{aligned} U_t + F(U)_x &= S(U,x) \quad \text{in } \mathbb{R} \times (0,T), \\ U(x,0) &= U_0(x) \quad \text{in } \mathbb{R}. \end{aligned} \tag{1}$$

Weak solutions: $U \in BV(\mathbb{R} \times (0,T))$ solves

$$\int_0^T \int_{\mathbb{R}} \Big(- U\,\varphi_t - F(U)\,\varphi_x + S(U)\,\varphi \Big) dxdt = \int_{\mathbb{R}} U_0 \varphi_0 \, dx \tag{2}$$

for any smooth, compactly supported test function φ.

Semidiscrete finite volume schemes:

$$\frac{d}{dt} U_K(t) + \frac{\mathcal{F}_R - \mathcal{F}_R}{\Delta x} = S_K. \tag{3}$$

Equilibrium variables

Chertock, Herty, Özcan 2017: equilibrium variables

$$V := F + R := F - \int^x S \tag{4}$$

Classical solutions:

$$U_t + V_x = 0. \tag{5}$$

Finite volume scheme:

$$\frac{d}{dt} U_K(t) + \frac{\mathcal{V}_R - \mathcal{V}_R}{\Delta x} = 0. \tag{6}$$

Advantage: reconstruction in V gives automatic **well-balancing**.

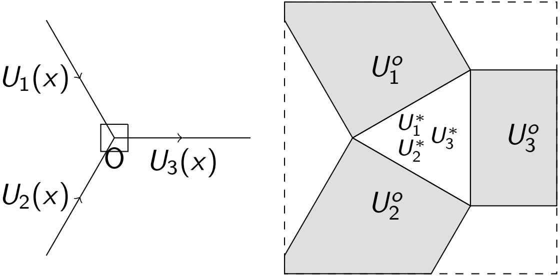
Outline

1. Advanced application: pipeline networks
   - Introduction
   - Coupling Conditions
   - Well-balanced Scheme
   - Numerical examples

2. Basic structure: conservation laws versus balance laws
   - Localized weak solutions
   - Semi-discrete limit
   - Equilibrium variables and one-sided fluxes

Pipeline networks

- 1D model for network of pipes



- U within the pipes given by Isothermal Euler equations

$$
\begin{aligned}
(\rho_i)_t + (q_i)_x &= 0 \\
(q_i)_t + \left(\frac{q_i^2}{\rho_i} + p(\rho_i) \right)_x &= -\frac{f_{g,i}}{2D_i} \frac{q_i|q_i|}{\rho_i}
\end{aligned}
\tag{7}
$$

Isothermal pressure $p(\rho) = a^2 \rho$

- Coupling conditions at node $\phi(U_1^*, U_2^*, ..., U_M^*) = 0$

- Schemes which preserve a steady state exactly are called well-balanced schemes
- Why do we need well-balanced schemes?
  - Nonlinear coupling conditions at the junction
  - Imbalance of flux and source terms

$$\cancelto{0}{U_t} + F(U)_x = S(U)$$

  - Leads to spurious oscillations for near equilibrium flows
- We extend the approach of Chertock, Herty, Özcan[2017] to model flow at the junctions
- Assumptions
  - Subsonic flow
  - Flow is unidirectional

Isothermal Euler equations

- Eigenvalues,

$$\lambda_1 = \frac{q}{\rho} - \sqrt{p'(\rho)} < 0$$
$$\lambda_2 = \frac{q}{\rho} + \sqrt{p'(\rho)} > 0$$

- Both characteristic fields are genuinely nonlinear

$$\nabla \lambda_i(U).r_i(U) = \pm \frac{a}{\rho} \neq 0$$

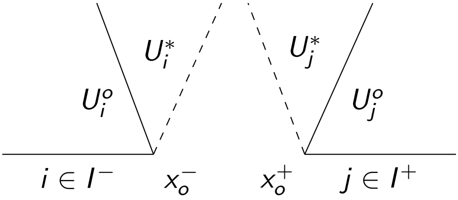


Figure: Conservative variables at the junction

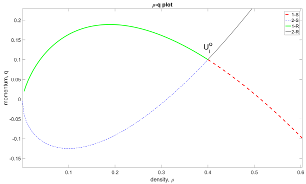


Figure: Phase plot for incoming pipe

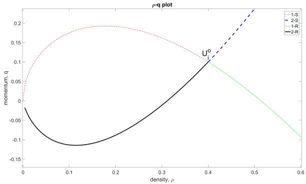


Figure: Phase plot for outgoing pipe

$$U_i^* = \bar{U}_i(\sigma_i; U_i^o)$$

Coupling Conditions

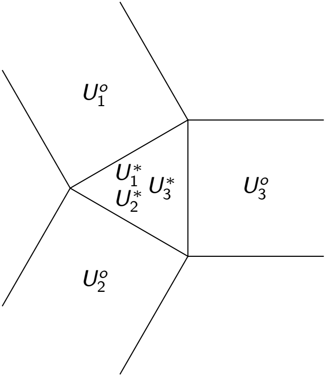
- Coupling conditions at junction given by Banda, Herty, Klar[2006];Herty, Seaid[2007]; etc
  - Mass balance at the junction

$$\sum_{i \in I^-} A_i q_i^* = \sum_{j \in I^+} A_j q_j^* \qquad (8)$$

  - Constant pressure at the junction

$$p(\rho_k^*) = p^* \quad \forall k \in I^- \cup I^+ \qquad (9)$$

- Existence and Uniqueness of solution for these coupling conditions given by Colombo, Garavello[2006]



For compressor

$$\begin{aligned} q_1^* &= q_2^* \\ p(\rho_2^*) &= CRp(\rho_1^*) \end{aligned} \qquad (10)$$

Well-balanced Scheme

- Equilibrium variables, V remain constant at steady state

$$U_t + V_x = 0, \tag{11}$$

$$V(U) = F(U) - \int^x S \tag{12}$$

- For isothermal Euler equations

$$\begin{aligned} (\rho_i)_t + (K_i)_x &= 0 \\ (q_i)_t + (L_i)_x &= 0 \end{aligned} \tag{13}$$

$$K_i = q_i, \quad L_i = \frac{q_i^2}{\rho_i} + p(\rho_i) + R_i(x),$$

$$R_i(x) = \int_{x_0}^{x} \frac{f_{g,i}}{2D_i} \frac{q_i|q_i|}{\rho_i}\, dx \tag{14}$$

Coupling conditions in terms of equilibrium variables

$$P(K, L, R) = \frac{L - R + \sqrt{(L - R)^2 - 4a^2K^2}}{2} \tag{15}$$

- Mass balance

$$\sum_{i \in I^-} A_i K_i^* = \sum_{j \in I^+} A_j K_j^* \tag{16}$$

- Constant pressure, p^* at junction

$$P(K_i^*, L_i^*, R_i^*) = p^* \tag{17}$$

- R_i at junction is constant, $R_i^* = R_i^o$

Similarly coupling conditions for compressor,

$$\begin{aligned} K_1^* &= K_2^* \\ P(K_2^*, L_2^*, R_2^*) &= CR\ P(K_1^*, L_1^*, R_1^*) \end{aligned} \tag{18}$$

Wave curves in terms of equilibrium variables

- The 1-wave curve for incoming pipe and 2-wave curve for outgoing pipe are monotonic in the subsonic region

$$V_i^* = \bar{V}_i(\sigma; V_i^o)$$

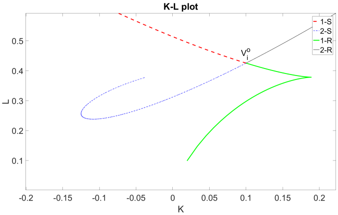


Figure: K-L plot for Lax curves of incoming pipe

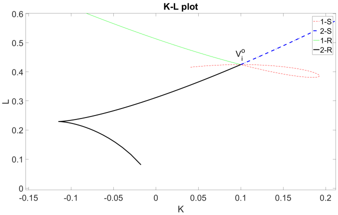


Figure: K-L plot for Lax curves of outgoing pipe

- The solution to the coupling conditions gives flux entering the pipes from the junction

Lemma

Consider a nodal point with $|I^-| \geq 1$ incoming and $|I^+| \geq 1$ outgoing adjacent pipes. $\widehat{V}_i = (\widehat{K_i}, \widehat{L_i}), i \in I^\pm$ *be the corresponding equilibrium variables, with integrated source terms* $\widehat{R}_i$.
Then there exists an open neighborhood $\mathcal{V} \subset \mathbb{R}^{2M \times M}$ *of* $(\widehat{V}, \widehat{R}) := (\widehat{V}_i, \widehat{R}_i)_{i \in I^\pm}$ *such that for any* $(V^o, R^o) \in \mathcal{V}$ *there exists a unique* V^* *such that* $(V^*, R^o) \in \mathcal{V}$ *fulfill the coupling conditions* (16) *and* (17).

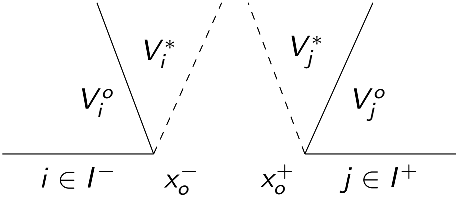


Figure: Equilibrium variables at the junction

Proof:

- We check the well-posedness of the coupling conditions in terms of equilibrium variables using approach of Colombo, Garavello[2006]
- Coupling Conditions

$$\Psi(V) = \begin{bmatrix} \sum_{i \in I^-} A_i K_i - \sum_{j \in I^+} A_j K_j \\ p(V_1) - p(V_2) \\ \vdots \\ p(V_{M-1}) - p(V_M) \end{bmatrix}$$

$$\Psi(\widehat{V}) = 0$$

$$D_\sigma \Psi = \begin{bmatrix} A_1 \frac{dK_1}{d\sigma_1} & \dots & |I^-| terms & -A_j \frac{dK_j}{d\sigma_j} & \dots & |I^+| terms \\ \frac{dp_1}{d\sigma_1} & -\frac{dp_2}{d\sigma_2} & 0 & \dots & \dots & 0 \\ 0 & \frac{dp_2}{d\sigma_2} & -\frac{dp_3}{d\sigma_3} & 0 & \dots & 0 \\ \vdots & & \ddots & \ddots & & \vdots \\ \vdots & & & \ddots & \ddots & 0 \\ 0 & \dots & \dots & 0 & \frac{dp_{M-1}}{d\sigma_{M-1}} & -\frac{dp_M}{d\sigma_M} \end{bmatrix}$$

$$
\begin{aligned}
det(D_\sigma \Psi) =& (-1)^{M-1} \sum_{i \in I^-} \left(A_i \frac{dK_i}{d\sigma_i} \prod_{k \in I^\pm, k \neq i} \frac{dp_k}{d\sigma_k} \right) \\
&+ (-1)^M \sum_{j \in I^+} \left(A_j \frac{dK_j}{d\sigma_j} \prod_{k \in I^\pm, k \neq j} \frac{dp_k}{d\sigma_k} \right)
\end{aligned}
$$

$$
\frac{dK_i}{d\sigma_i}(\sigma_i = 0) = \begin{cases} < 0 & \forall i \in I^- \\ > 0 & \forall i \in I^+ \end{cases}, \quad \frac{dp_i}{d\sigma_i}(\sigma_i = 0) > 0
$$

$$
det(D_\sigma \Psi) \neq 0
$$

Thus by IFT there exists neighborhood $\mathcal{V}$ for the point $(\widehat{V}_i, \widehat{R}_i)$, such that for all initial data (V^o, R^o), the solution to the coupling condition exists and and converges to the steady state $(\widehat{V}_i, \widehat{R}_i)$.

□

Well-balanced scheme

- Central Upwind Scheme

$$\frac{dU_i^j}{dt} = -\frac{\mathcal{V}_i^{j+1/2} - \mathcal{V}_i^{j-1/2}}{\Delta x} \tag{19}$$

- At the junction

$$\begin{aligned} \mathcal{V}_i^{N+1/2} &= V_i^*, \ i \in I^- \\ \mathcal{V}_i^{1/2} &= V_i^*, \ i \in I^+ \end{aligned} \tag{20}$$

with $V_i^o = V_i^{N,E}, i \in I^-$ and $V_i^o = V_i^{1,W}, i \in I^+$

- $V_i^{j,E/W}$ are piecewise linear reconstruction for V_i^j

$$V_i^{j,E} = V_i^j + \frac{\Delta x}{2}(V_x)_i^j, \ V_i^{j,W} = V_i^j - \frac{\Delta x}{2}(V_x)_i^j \tag{21}$$

$$(V_x)_i^j = \textbf{minmod}\left(\theta \frac{V_i^{j+1} - V_i^j}{\Delta x}, \frac{V_i^{j+1} - V_i^{j-1}}{2\Delta x}, \theta \frac{V_i^j - V_i^{j-1}}{\Delta x}\right), \quad \theta \in [1,2] \tag{22}$$

- Minmod function is defined as,

$$\textbf{minmod}(v_1, v_2, \ldots, v_n) = \begin{cases} min(v_1, v_2, \ldots, v_n) & \text{if } v_i > 0 \forall i \\ max(v_1, v_2, \ldots, v_n) & \text{if } v_i < 0 \forall i \\ 0 & \text{otherwise} \end{cases} \tag{23}$$

Well-balanced scheme

- The integral of source term is calculated using second-order quadrature with $R_i^{1/2} = R_k^{N+1/2} = 0 \quad \forall i \in I^+, k \in I^-$

$$R_i^{j+1/2} = R_i^{j-1/2} + \Delta x \frac{f_{g,i}}{2D_i} \frac{q_i^j |q_i^j|}{\rho_i^j}, \; R_k^{j-1/2} = R_k^{j+1/2} + \Delta x \frac{f_{g,k}}{2D_k} \frac{q_k^j |q_k^j|}{\rho_k^j}. \quad (24)$$

- Flux for interior cell boundaries of each pipe is same as that used by Chertock, Herty, Özcan[2017]

$$\mathcal{V}_i^{j+1/2} = \frac{a_{i,+}^{j+1/2} V_i^{j,E} - a_{i,-}^{j+1/2} V_i^{j+1,W}}{a_{i,+}^{j+1/2} - a_{i,-}^{j+1/2}} + \alpha_i^{j+1/2} (U_i^{j+1,W} - U_i^{j,E}) \mathcal{H}\Big(\frac{|V_i^{j+1} - V_i^j|}{\Delta x} \frac{|\Omega|}{\max_j \{V_i^j\}} \Big) \quad (25)$$

$$\mathcal{H}(\phi) = \frac{(C\phi)^m}{1 + (C\phi)^m}, \quad C, m > 0$$

- $a_{i,+}^{j+1/2}, a_{i,-}^{j+1/2}$ are maximum and minimum eigenvalues respectively and $\alpha_i^{j+1/2} = \frac{a_{i,+}^{j+1/2} a_{i,-}^{j+1/2}}{a_{i,+}^{j+1/2} - a_{i,-}^{j+1/2}}$

Lemma

The numerical scheme given by (19) *and flux defined by* (25) *preserves the steady state across a node of M adjacent pipes and coupling conditions given by* (16) *and* (17).

Proof:

- Steady state defined by constant flux within each pipe and satisfying coupling condition at junction
- The definition of the numerical fluxes in (25) ensure equilibrium variables are constant in each pipe
- From previous lemma, coupling conditions have unique solution

Steady state at junction of pipes

Initial conditions

- 1 incoming, 1 outgoing pipe $K_1 = K_2 = 0.15$ and $L_1 = L_2 = 0.4$
- 1 incoming, 2 outgoing pipe $K_1 = 0.15, K_2 = K_3 = 0.075$ and $p^* = 0.332$ or $L_1 = 0.4, L_2 = L_3 = 0.3492$
- 2 incoming, 1 outgoing pipe $K_3 = 0.15, K_1 = K_2 = 0.075$ and $p^* = 0.332$ or $L_3 = 0.4, L_1 = L_2 = 0.3492$

Table: Comparison of L-1 errors between well-balanced(WB) and non well-balanced(NWB) scheme at steady state for a junction at time T=1

No. of cells in each pipe	L1-error for variable	1 Incoming, 1 Outgoing		1 Incoming, 2 Outgoing		2 Incoming, 1 Outgoing	
		WB	NWB	WB	NWB	WB	NWB
100	K	8.74×10^{-17}	1.56×10^{-7}	1.30×10^{-16}	9.63×10^{-8}	1.14×10^{-16}	8.67×10^{-8}
	L	1.27×10^{-16}	2.43×10^{-7}	7.74×10^{-17}	8.94×10^{-8}	8.30×10^{-17}	1.87×10^{-7}

Steady state for compressor

Initial condition
$K_1 = K_2 = 0.15$ and $p_1^* = 0.332, p_2^* = CRp_1^*$

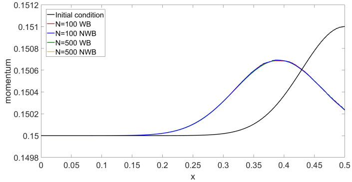
Table: Comparison of L-1 errors between well-balanced(WB) and non well-balanced(NWB) scheme at steady state with a compressor at different compression ratios at time T=1

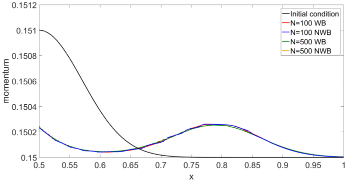
No. of cells in each pipe	L1-error for variable	CR=1.5		CR=2.0		CR=2.5	
		WB	NWB	WB	NWB	WB	NWB
100	K	3.91×10^{-17}	1.05×10^{-7}	1.30×10^{-16}	9.63×10^{-8}	4.72×10^{-16}	9.68×10^{-8}
	L	5.59×10^{-16}	1.01×10^{-7}	7.74×10^{-17}	8.94×10^{-8}	3.61×10^{-17}	8.89×10^{-7}

1 incoming, 1 outgoing pipes

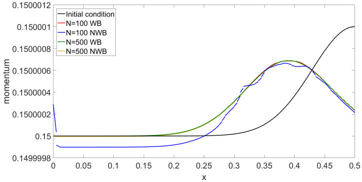
Initial condition $K_i = K_i^* + \eta_i e^{-100(x-0.5)^2}, \quad L_i = L_i^*$

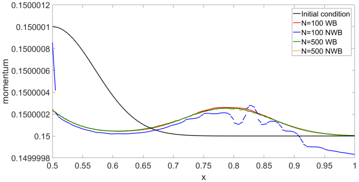
$K_i^* = 0.15,\ L_i^* = 0.4,\ \eta = 10^{-3}$





$\eta = 10^{-6}$



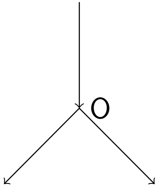


(a) Pipe 1 (b) Pipe 2

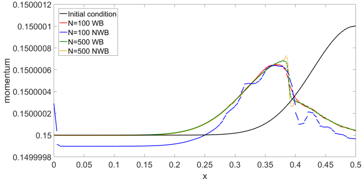
1 incoming, 2 outgoing pipes

Initial condition $K_i = K_i^* + \eta_i e^{-100(x-0.5)^2}, \quad L_i = L_i^*$

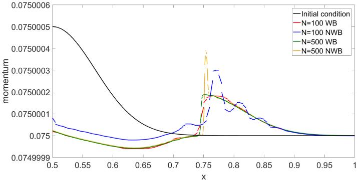
$K_1^* = 0.15, K_2^* = K_3^* = 0.075, \; \eta_1^* = 10^{-6}, \eta_2^* = \eta_3^* = 0.5\text{x}10^{-6}$



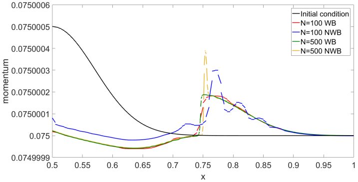
(a) Junc. with 1 incoming and 2 outgoing pipe



(b) Pipe 1



(c) Pipe 2



(d) Pipe 3

Summary:

- Equilibrium and near equilibrium flows are resolved accurately for a junction of gas pipelines.

Work in progress:

- more complex networks
- higher order DG
- study of energy dissipation and entropy production

References

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- R.Colombo, and M. Garavello. A well posed Riemann problem for the p-system at a junction. *Netw. Heterog. Media*, 2006.
- A. Chertock, M. Herty, and S. Özcan. Well-balanced central-upwind schemes for 2×2 systems of balance laws. *Proceedings of the XVI International Conference on Hyperbolic Problems*, Springer(accepted), 2017.
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2. Basic structure: conservation laws versus balance laws
   - Localized weak solutions
   - Semi-discrete limit
   - Equilibrium variables and one-sided fluxes

Solutions to the classical balance law

Classical solutions:

$$U_t + F_x = S \tag{26}$$

Weak solutions:

$$\int_0^T \int_{\mathbb{R}} \Big(- U\,\varphi_t - F\,\varphi_x + S\,\varphi \Big) dxdt = \int_{\mathbb{R}} U_0 \varphi_0 \, dx \tag{27}$$

Localization

Space-time cell

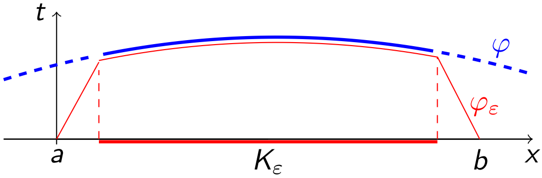
$$K := (a, b) \times (0, \Delta t).$$

Interior cell: $\varepsilon \ll \Delta t$

$$K_\varepsilon := \{(x,t) \in K \mid \operatorname{dist}\big((x,t), \partial K\big) > \varepsilon\}$$

Cut-off test function: $\varphi_\varepsilon \in C_0^1(\bar{K})$ such that

$$\varphi_\varepsilon(x,t) = \begin{cases} \varphi(x,t), & \text{for } (x,t) \in K_\varepsilon \quad \text{(interior of cell)} \\ 0, & \text{for } (x,t) \notin K. \end{cases}$$



For a piecewise smooth weak solution,

$$0 = \iint_{K_\varepsilon} \Big(- U\,\varphi_t - F\,\varphi_x + S\varphi \Big) dxdt + \iint_{K \setminus K_\varepsilon} \Big(- U\,\varphi_{\varepsilon,t} - F\,\varphi_{\varepsilon,x} + S\varphi \Big) dxdt \tag{28}$$

As $\varepsilon \to 0$, the integral over the boundary strip,

$$- \iint_{K \setminus K_\varepsilon} \Big(\varphi_{\varepsilon,t}, \varphi_{\varepsilon,x} \Big)(\cdot, \cdot) dxdt$$

becomes a Dirac measure and we obtain

Theorem (localized weak solution) Let U be a p.w. smooth weak solution and φ a (globally defined) test function. Then, for any subcell,

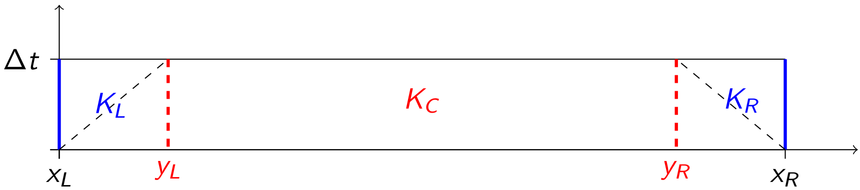
$$0 = \int_0^{\Delta t} \int_a^b \Big(- U\,\varphi_t - F\,\varphi_x + S\,\varphi \Big) dxdt + \int_a^b \hat{U}_K\,\varphi\big|_{t=0}^{\Delta t}\,dx + \int_0^{\Delta t} \hat{F}_K\,\varphi\big|_{x=a}^{b}\,dt. \qquad (29)$$

where $\hat{U}_K$ and $\hat{F}_K$ are interior traces w/r cell K.

Semi-discrete limit

$[x_L, x_R] \times [0, \Delta t]$	grid cell
$s := \max_K \rho(F'(u))$	maximal wave speed
$y_L := x_L + s\Delta t$	$y_R := x_R - s\Delta t$

Consider the domain $K = K_L \cup K_C \cup K_R$



Consider K_C. Divide (29) by Δt:

$$
\begin{aligned}
0 = \frac{1}{\Delta t} \int_0^{\Delta t} \int_{y_L}^{y_R} \Big(- U\varphi_t - F\varphi_x + S\varphi \Big) dxdt \\
+ \int_{y_L}^{y_R} \frac{\hat{U}(\Delta t) - \hat{U}(0)}{\Delta t} \varphi \, dx \\
+ \frac{\varphi(x_R, 0)}{\Delta t} \int_0^{\Delta t} F(U(y_R)) \, dt - \frac{\varphi(x_L, 0)}{\Delta t} \int_0^{\Delta t} F(U(y_L)) \, dt \\
+ \mathcal{O}(\Delta t). \qquad (30)
\end{aligned}
$$

For K_L,

$$0 = \int_{x_L}^{y_L} \frac{\hat{U}(\Delta t) - \hat{U}(0)}{\Delta t}\, dx + \frac{1}{\Delta t}\int_0^{\Delta t} F(U(y_L))\, dt - \frac{1}{\Delta t}\int_0^{\Delta t} \hat{F}(U(x_L))\, dt + \mathcal{O}(\Delta t). \tag{31}$$

Similarly for K_R.

Classical finite volume scheme

Add the weak formulations over K_L, K_C, K_R, let $\varphi = \varphi(x)$ and pass to the limit:

$$0 = \lim_{\Delta t \to 0} \int_{x_L}^{x_R} \frac{\hat{U}(\Delta t) - \hat{U}(0)}{\Delta t} \varphi \, dx + \Big(\varphi(x) \, \widehat{F}(U(x,t)) \Big) \Big|_{x=x_L}^{x_R} + \int_{x_L}^{x_R} \Big(-F\varphi_x + S\varphi \Big) dx \qquad (32)$$

Due to the Rankine-Hugoniot condition, the flux is the solution of the Riemann problem at the interface.,

$$\widehat{F}(U(x_L, t)) = \mathcal{F}_L = \mathcal{F}_{\text{Riem}}(U(x_L-), U(x_L+)). \qquad (33)$$

One-sided equilibrium fluxes

Similarly, in (U, V) variables,

$$0 = \lim_{\Delta t \to 0} \int_{x_L}^{x_R} \frac{\hat{U}(\Delta t) - \hat{U}(0)}{\Delta t} \varphi \, dx$$
$$+ \left(\varphi(x) \, \widehat{V}(U, x) \right) \Big|_{x=x_L}^{x_R} + \int_{x_L}^{x_R} \left(- V \varphi_x \right) dx \qquad (34)$$

However,

$$\widehat{V}(U, x_L) = \mathcal{F}_L + \widehat{R}_L^{+} =: \widehat{V}_L^{+} \qquad (35)$$
$$\widehat{V}(U, x_R) = \mathcal{F}_L + \widehat{R}_R^{-} =: \widehat{V}_R^{-} \qquad (36)$$

Finite volume updates

Traditional update:

$$\frac{d}{dt}U_K(t) = -\frac{\mathcal{F}_R - \mathcal{F}_L}{\Delta x} + S_K \tag{37}$$

Chertock et al. update:

$$\frac{d}{dt}U_K(t) = -\frac{\widehat{V}_R^- - \widehat{V}_L^+}{\Delta x} \tag{38}$$

Possible advantages of one-sided equilibrium fluxes

- simplify numerical flux (see pp. 17 - 18)
- a new look on reconstructions
- pipeline networks
- multi-D

$$\operatorname{div} R = S.$$